Vectors and Matrices

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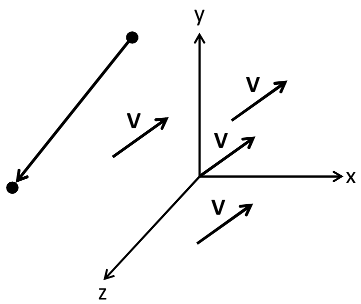
Basic Entities

- Coordinate system: has an origin and some mutually perpendicular axes emanating from the origin
- Point P: a location in space
- Vector **V** : a directed line segment that has magnitude (length) and direction, e.g. physical entities such as force and velocity
  - Vector has no fixed location, seen as points displacement



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Vectors

- Vectors defined by *n*-tuples of real numbers, where *n* is typically 2, 3, or 4. An *n*-dimensional vector **V** can be written as
$$\boldsymbol{V} = \langle V_1, V_2, \dots, V_n \rangle$$
  - where the numbers V_i are called the *components* of **V**
- In 3D, $\boldsymbol{V} = \langle V_x, V_y, V_z \rangle$
- The vector **V** may also be represented by a matrix having a single column and *n* rows:
$$\boldsymbol{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$$
- We write row vectors as the transpose of their corresponding column vectors as $\boldsymbol{V^T} = [V_1 \quad V_2 \quad \dots \quad V_n]$

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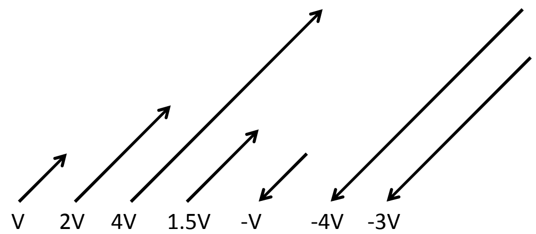
Multiply Vectors by a Scalar

- A vector may be multiplied by a scalar to produce a new vector whose components retain the same relative proportions
$$\boldsymbol{aV} = \langle aV_1, aV_2, \dots, aV_n \rangle$$
- If *a* = −1, we use notation −**V** to represent the negation of the vector **V**



Geometric Interpretation

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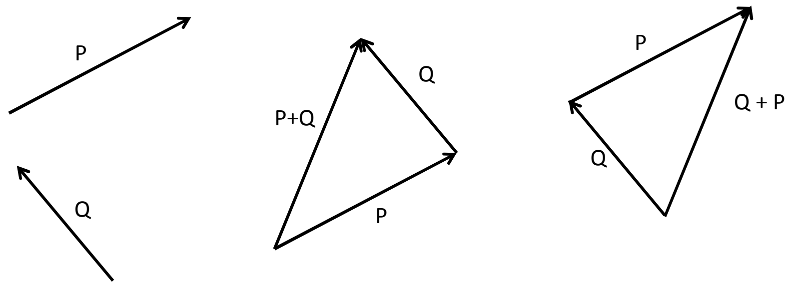
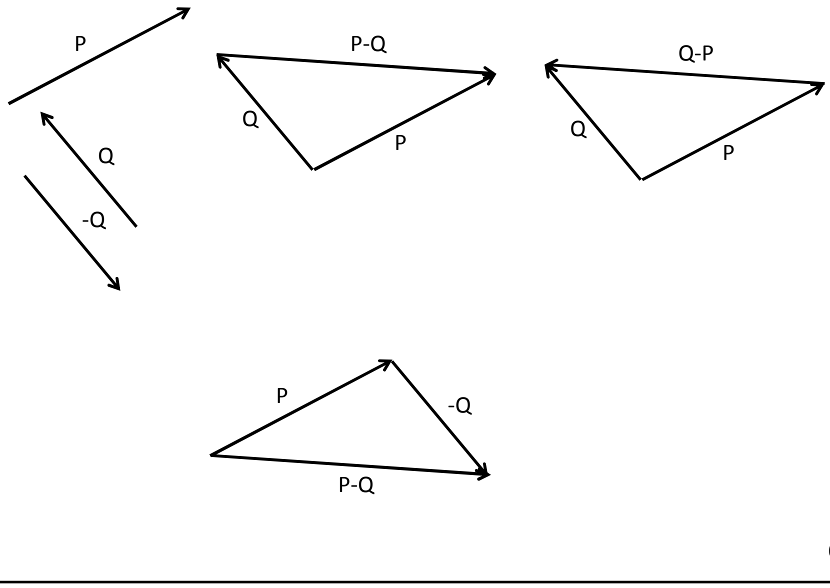
Vectors Addition and Subtraction

- Given two vectors **P** and **Q**, we define the sum **P** + **Q** as

  P + Q$= \langle P_1 + Q_1, P_2 + Q_2, \ldots, P_n + Q_n \rangle$
- **P - Q** is really just a notational simplification of the sum **P** + (−**Q**)
- Geometrically, we can add vectors **P** and **Q** geometrically by positioning the vectors so that the head of **P** touches the tail of **Q**, and then draw a vector from the tail of **P** to the head of **Q**.



Vectors Addition and Subtraction

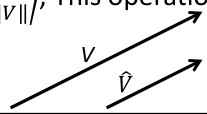


Vectors Magnitude (Length)

- The *magnitude* of an *n*-dimensional vector **V** is a scalar denoted by $\|\boldsymbol{V}\|$ and is given by the formula

$$\|V\| = \sqrt{\sum_{i=1}^{n} {V_i}^2}$$

- In 3D, $\|V\| = \sqrt{{V_x}^2 + {V_y}^2 + {V_z}^2}$
- A vector having a magnitude of exactly one is called a *unit vector*
- A vector **V** having at least one nonzero component can be resized to unit length through multiplication by $\frac{1}{\|V\|}$
- $\hat{V} = \frac{V}{\|V\|} = \left\langle \frac{V_1}{\|V\|}, \frac{V_2}{\|V\|}, \dots, \frac{V_n}{\|V\|} \right\rangle$, This operation is called *normalization*



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Example

Given a vector V = $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ then 2V = $\begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix}$

Given vector r = $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and s = $\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

r + s = $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ + $\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ = $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$

r - s = $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ + -1 * $\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ = $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ + $\begin{bmatrix} -4 \\ -1 \\ -1 \end{bmatrix}$ = $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

$\|r\| = \sqrt{1^2 + 2^2 + 1^2}$ = $\sqrt{6}$, $\hat{r}$ = $\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

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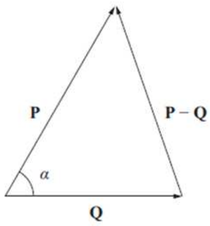
Dot Product

- The dot product of two n-dimensional vectors **P** and **Q**, written as **P** ·**Q**, is the scalar quantity given by the formula

$$P.Q = \sum_{i=1}^{n} P_i Q_i$$

- In 3D, $P.Q = P_x Q_x + P_y Q_y + P_z Q_z$
- Geometrically, the dot product in any dimension tells how "similar" two vectors are; the larger the dot product, the more similar the two vectors



- The dot product **P** ·**Q** satisfies the equation

$$P.Q = \|P\| \|Q\| \cos \alpha$$

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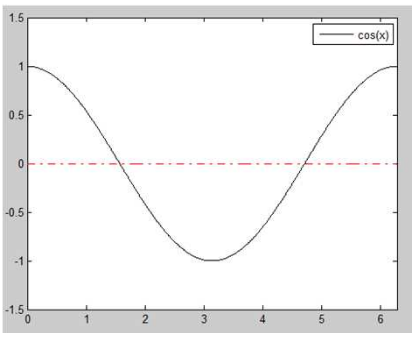
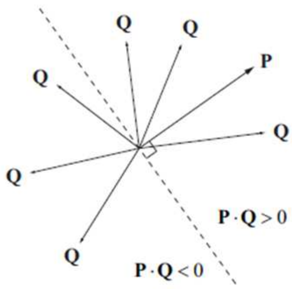
Dot Product

- Two vectors **P** and **Q** are perpendicular if and only if **P** ·**Q** = 0
- Any vector lying on the same side of the plane as **P** yields a positive dot product with **P**, and any vector lying on the opposite side of the plane from **P** yields a negative dot product with **P**





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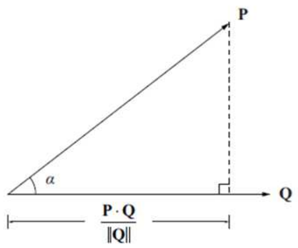
Vector Projection

- The projection of **P** onto the vector **Q** produces the side adjacent to the angle α between **P** and **Q**
- The length of the side adjacent to α is given by $\|P\| \cos\alpha = \frac{P.Q}{\|Q\|}$



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Example

Given vector r = $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ and s = $\begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix}$

$\|r\|$ = $\sqrt{2^2 + (-3)^2 + 4^2}$ = $\sqrt{29}$ = 5.385

$\|s\|$ = $\sqrt{5^2 + 6^2 + 10^2}$ = $\sqrt{161}$ = 12.689

$\|r\|\ \|s\|$ cos α = 2 * 5 + (-3) * 6 + 4 * 10 = 32

5.385 * 12.689 * cos α = 32

cos α = 32 / (5.385 * 12.689) = 0.468

α = $\cos^{-1}$(0.468) = 62.1

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Cross Product

- The *cross product* of two three-dimensional vectors, also known as the *vector product*, returns a new vector that is perpendicular to both of the vectors being multiplied together

$$P \times Q = \langle P_y Q_z - P_z Q_y, P_z Q_x - P_x Q_z, P_x Q_y - P_y Q_x \rangle$$

- A commonly used tool for remembering this formula is to calculate cross products by evaluating the pseudodeterminant

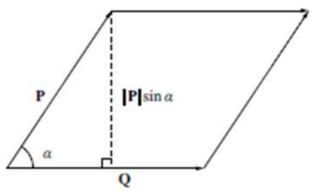
$$P \times Q = \begin{vmatrix} i & j & k \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$
$$= i(P_y Q_z - P_z Q_y) - j(P_x Q_z - P_z Q_x) + k(P_x Q_y - P_y Q_x)$$

- where i, j, and k are unit vectors parallel to the *x*, *y*, and *z* axes:
- $i = \langle 1,0,0 \rangle, j = \langle 0,1,0 \rangle$ and $k = \langle 0,0,1 \rangle$

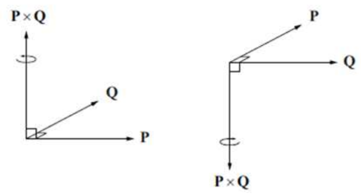
Cross Product

- The cross product **P** ×**Q** satisfies the equation

$$\|P \times Q\| = \|P\| \|Q\| \sin \alpha$$



- The cross product follows the *right hand rule*. If the fingers of the right hand are aligned with a **P**, and the palm is facing in the direction of a **Q**, then the thumb points to the direction of **P** ×**Q**.



Example: Unit Normal Vector for a Triangle

r $= P_1 - P_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and s $= P_2 - P_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

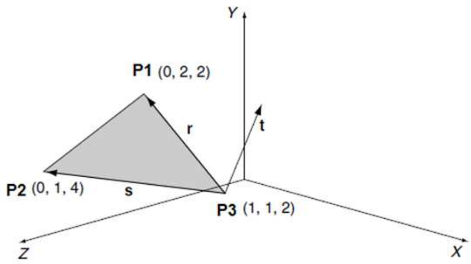
r x s $= \begin{bmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

r x s $= \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ = t

$\|t\| = \sqrt{2^2 + 2^2 + 1^2}$ = 3

t is not a unit vector

$\hat{t}$ = $\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$



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Matrices

- An *n* × *m* matrix **M** is an array of numbers having *n* rows and *m* columns. If *n* = *m*, then we say that the matrix **M** is *square*

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

- Consider a matrix **M** with dimensions *n* × *m*. The *transpose* of **M** (denoted $\mathbf{M}^T$) is the *m* × *n* matrix where the columns are formed from the rows of **M**. In other words, $\mathbf{M}^T_{ij} = \mathbf{M}_{ji}$

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 5 & 6 \\ 8 & 9 & 10 \\ 15 & 20 & 25 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 8 & 15 \\ 3 & 5 & 9 & 20 \\ 7 & 6 & 10 & 25 \end{bmatrix}$$

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Matrix Multiplication

- Given a scalar *a* and an *n*×*m* matrix **M**, the product *a***M** is given by

$$aM = \begin{bmatrix} aM_{11} & aM_{12} & aM_{13} \\ aM_{21} & aM_{22} & aM_{23} \\ aM_{31} & aM_{32} & aM_{33} \end{bmatrix}$$

- Two matrices **F** and **G** can be multiplied together, provided that the number of columns in **F** is equal to the number of rows in **G**. If **F** is an *n* × *m* matrix and **G** is an *m*× *p* matrix, then the product **FG** is an *n*× *p* matrix whose (*i*, *j*) entry is given by

$$(FG)_{ij} = \sum_{k=1}^{m} F_{ik} G_{kj}$$

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Matrix Multiplication

- $A = \begin{bmatrix} 1 & -5 & 3 \\ 0 & -2 & 6 \\ 7 & 2 & -4 \end{bmatrix}, B = \begin{bmatrix} -8 & 6 & 1 \\ 7 & 0 & -3 \\ 2 & 4 & 5 \end{bmatrix}$
- $AB = \begin{bmatrix} -37 & 18 & 31 \\ -2 & 24 & 36 \\ -50 & 26 & -19 \end{bmatrix}$
- There is an *n* × *n* matrix called the *identity* matrix, denoted by $\mathbf{I}_n$, for which $\mathbf{MI}_n = \mathbf{I}_n\mathbf{M} = \mathbf{M}$ for any *n* × *n* matrix **M**

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- Matrix multiplication is *not* commutative: $AB \neq BA$
- Matrix multiplication is associative: $(AB)C = A(BC)$

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Multiplying a Vector and a Matrix

- Row vectors are multiplied on the left, while column vectors are multiplied on the right

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$= \begin{bmatrix} xM_{11} + yM_{21} + zM_{31} & xM_{12} + yM_{22} + zM_{32} & xM_{13} + yM_{23} + zM_{33} \end{bmatrix}$$

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} xM_{11} + yM_{12} + zM_{13} \\ xM_{21} + yM_{22} + zM_{23} \\ xM_{31} + yM_{32} + zM_{33} \end{bmatrix}$$

- We adopt column vector representation. WHY?

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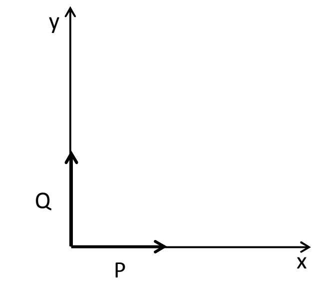
How does a Matrix Transform Vectors?

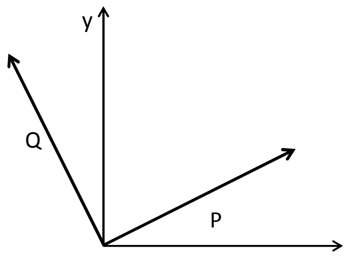
- Matrix performs vectors transformation. If **Ma**=**b**, we say that **M** transformed **a** to **b**

Let $M = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, vectors $P = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$





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