Elementary

Graph Algorithms

PART-2

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Outlines: Graphs Part-2

- Graph Search Methods
- Breadth-First Search (BFS):
  - **BFS Algorithm**
  - **BFS Example**
  - **BFS Time Complexity**
  - **Output of BFS:**
    - **Shortest Path**
    - **Breath-First Tree**
    - **Is the Graph Connected?**

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Graph Search Methods

- Many graph problems solved using a search method
  - Path from one vertex to another
  - Is the graph connected?
  - etc.
- Commonly used search methods:
  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)

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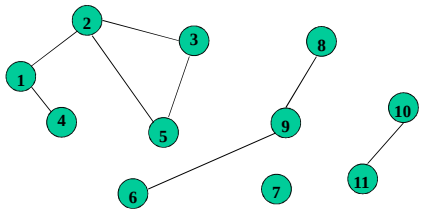
Graph Search Methods

- A vertex u is reachable from vertex v iff there is a path from v to u.
- A search method starts at a given vertex v and visits every vertex that is reachable from v.



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Graph Search Methods

- Given: a graph G = (*V*, *E*), **directed** or **undirected**
- Goal: methodically explore **every vertex** and **every edge**
- Ultimately: build a **tree** on the graph
  - Pick a vertex as the root
  - Choose certain edges to produce a ***tree***
  - Note: might also build a ***forest*** if graph is not connected

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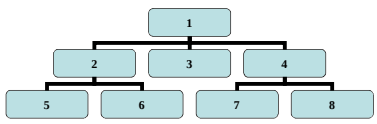
Breadth First Search

- Example: Binary Tree (This is a special case of a graph).
  - The order of search is across levels.
  - The root is examined first; then children of the root; then the children of those nodes, and so on.



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Breadth First Search

- Example: Directed Graph
- Pick a source vertex ***S*** to start.
- Find (or discover) the vertices that are adjacent to ***S***.
- Pick each child of ***S*** in turn and discover their vertices adjacent to that child.
- Done when all children have been discovered and examined.
- This results in a tree that is rooted at the source vertex ***S***.
- The idea is to find the *distance* from the selected Source vertex to all reachable vertices (Destinations).

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Breadth-First Search

- Visit start vertex (s) and put into a **FIFO** queue.
- Repeatedly remove a vertex from the queue, visit its unvisited adjacent vertices, put newly visited vertices into the queue.
- All vertices reachable from the start vertex (s) (including the start vertex) are visited.

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Breadth-First Search

- Again will associate vertex "colors" to guide the algorithm
  - **White** vertices have not been discovered
    - All vertices start out white
  - **Gray** vertices are discovered but not fully explored
    - They may be adjacent to white vertices
  - **Black** vertices are discovered and fully explored
    - They are adjacent only to black and gray vertices
- Explore vertices by scanning adjacency list of gray vertices

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Breadth-First Search

```
BFS(G, s) {
    // initialize vertices;
1   for each u ∈ V(G) – {s}{
2       do color[u] = WHITE
3          d[u] = ∞              // distance from s to u
4          p[u] = NIL            // predecessor or parent of u
    }
5   color[s] = GRAY
6   d[s] = 0
7   p[s] = NIL
8   Q = Empty;
9   Enqueue (Q,s);               // Q is a queue; initialize to s
10  while (Q not empty) {
11      u = Dequeue(Q);
12      for each v ∈ adj[u] {
13          if (color[v] == WHITE)
14              color[v] = GRAY;
15              d[v] = d[u] + 1;
16              p[v] = u;                 What does d[v] represent?
17              Enqueue(Q, v);            What does p[v] represent?
        }
18      color[u] = BLACK;
    }
}
```

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Breadth-First Search

- Lines 1-4 paint every vertex white, set d[u] to be infinity for each vertex (u), and set p[u] the parent of every vertex to be NIL.
- Line 5 paints the source vertex (s) gray.
- Line 6 initializes d[s] to 0.
- Line 7 sets the parent of the source to be NIL.
- Lines 8-9 initialize Q to the queue containing just the vertex (s).
- The while loop of lines 10-18 iterates as long as there remain gray vertices, which are discovered vertices that have not yet had their adjacency lists fully examined.
  - This while loop maintains the test in line 10, the queue Q consists of the set of the gray vertices.

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Breadth-First Search

- Prior to the first iteration in line 10, the only gray vertex, and the only vertex in Q, is the source vertex (s).
- Line 11 determines the gray vertex (u) at the head of the queue Q and removes it from Q.
- The for loop of lines 12-17 considers each vertex (v) in the adjacency list of (u).
- If (v) is white, then it has not yet been discovered, and the algorithm discovers it by executing lines 14-17.
  - It is first grayed, and its distance d[v] is set to d[u]+1.
  - Then, u is recorded as its parent.
  - Finally, it is placed at the tail of the queue Q.
- When all the vertices on (u's) adjacency list have been examined, u is blackened in line 18.

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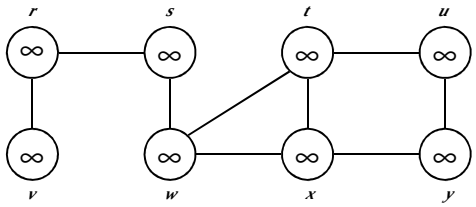
Breadth-First Search: Example



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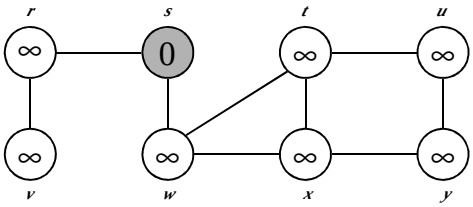
Breadth-First Search: Example



Q: s

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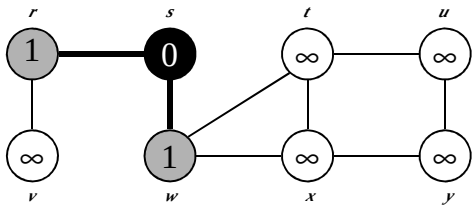
Breadth-First Search: Example



Q: w r

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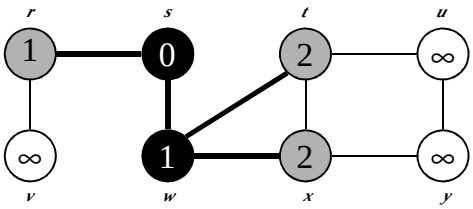
Breadth-First Search: Example



Q: r t x

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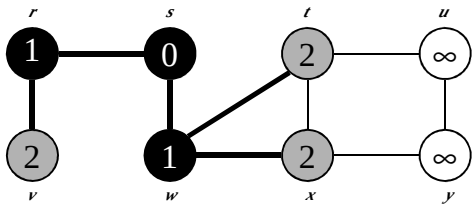
Breadth-First Search: Example



Q: t x v

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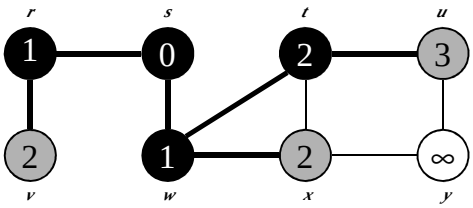
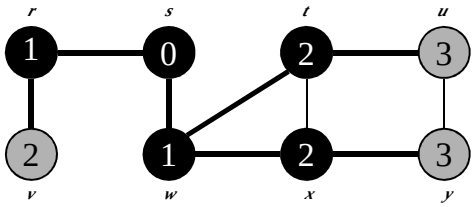
Breadth-First Search: Example



Q: x v u

18

Breadth-First Search: Example



Q: | v | u | y |

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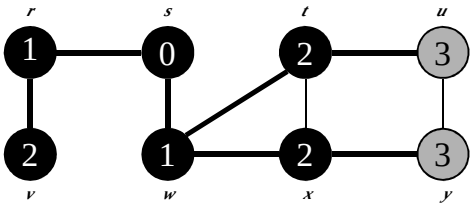
Breadth-First Search: Example



Q: | u | y |

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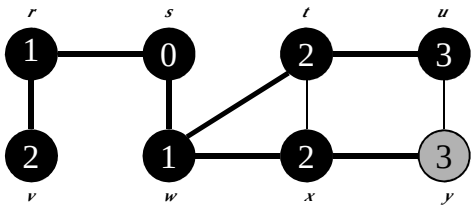
Breadth-First Search: Example



Q: | y |

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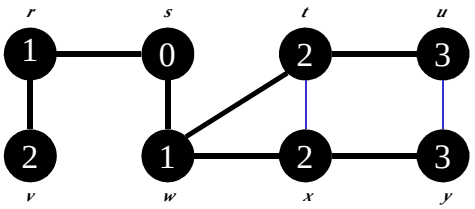
Breadth-First Search: Example



Q: Ø

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BFS: The Code Again

```
BFS(G, s) {
    // initialize vertices
    for each u ∈ V(G) – {s}{
        do color[u] = WHITE
           d[u] = ∞
           p[u] = NIL
    }
    color[s] = GRAY;
    d[s] = 0;
    p[s] = NIL;
    Q = Empty;
    Enqueue (Q,s);
    while (Q not empty) {
        u = Dequeue(Q);
        for each v ∈ adj[u] {
            if (color[v] == WHITE)
                color[v] = GRAY;
                d[v] = d[u] + 1;
                p[v] = u;
                Enqueue(Q, v);
        }
        color[u] = BLACK;
    }
}
```

Touch every vertex: $O(V)$

u = every vertex, but only once

So v = every vertex that appears in some other vert's adjacency list — *$O(E)$*

What will be the running time?

Total running time: $O(V+E)$

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BFS: Time Complexity

- Given a graph $G = (V, E)$
  - Vertices are enqueued if there color is white
  - Assuming that en- and dequeuing takes $O(1)$ time the total cost of this operation is $O(V)$
  - Adjacency list of a vertex is scanned when the vertex is dequeued (and at most once)
  - The sum of the lengths of all lists is $O(E)$. Consequently, $O(E)$ time is spent on scanning them
  - Initializing the algorithm takes $O(V)$
- **Total running time $O(V+E)$**
  - linear in the size of the adjacency list representation of G

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BFS: The Code Again

```
BFS(G, s) {
    // initialize vertices
    for each u ∈ V(G) – {s}{
       do color[u] = WHITE
          d[u] = ∞
          p[u] = NIL
    }
    color[s] = GRAY
    d[s] = 0
    p[s] = NIL
    Q = Empty;
    Enqueue (Q,s);
    while (Q not empty) {
        u = Dequeue(Q);
        for each v ∈ adj[u] {
            if (color[v] == WHITE)
                color[v] = GREY;
                d[v] = d[u] + 1;
                p[v] = u;
                Enqueue(Q, v);
        }
        color[u] = BLACK;
    }
}
```

Each vertex will be enqueue once.

What will be the storage cost in addition to storing the tree?

Queue Size

Total space used:
at least = $O(\max(\text{degree}(v)))$
at most = $O(V)$

Shortest Paths

- Given a graph $G = (V, E)$, BFS **discovers all vertices reachable from a source vertex s**
  - It computes the ***shortest-path distance*** to all reachable vertices
  - It computes a ***breadth-first tree*** that contains all such reachable vertices
- For any vertex v reachable from s, the path in the breadth first tree from s to v, corresponds to a **shortest path** in G.
- BFS calculates the ***shortest-path distance*** to the source node
  - shortest-path distance = minimum number of edges from s to $v = \delta(s, v) = d[v]$
  - or $\delta(s, v) = \infty$ if v not reachable from s

Breadth-First Tree

- BFS builds *breadth-first tree*, in which paths to root represent shortest paths in G
- Given a graph $G = (V, E)$ with source $s \in V$, we define the prdecessor subgraph of G as $G_p = (V_p, E_p)$, where
  - $V_p = \{v \in V : p[v] \neq \text{NIL}\} \cup \{s\}$
  - $E_p = \{(p[v], v) : v \in V_p - \{s\}\}$
- G_p is a breadth-first tree
  - V_p consists of the vertices reachable from s, and
  - for all $v \in V_p$, there is ***a unique simple path*** from s to v in G_p that is also a shortest path from s to v in G.
- The edges in G_p are called ***tree edges***

Path From Vertex v To Vertex u

- Start a breadth-first search at vertex v.
- ***Terminate*** when vertex u is visited or when Q becomes empty (whichever occurs first).
- Time $O(V+E)$

Is The Graph Connected?

- Start a breadth-first search at any vertex of the graph.
- Graph is connected iff all n vertices get visited.
- Time $O(V+E)$