Elementary

Graph Algorithms

PART-3

Outlines: Graphs Part-3

- Depth-First Search:
  - **DFS Algorithm**
  - **DFS Example**
  - **DFS Running Time**
  - **DFS Predecessor Subgraph**
  - **DFS Time Stamping**
  - **DFS Parenthesis Theorem**
  - **DFS Edge Classification**
  - **DFS and Graph Cycles**

Depth-First Search

- *Depth-first search* is another strategy for exploring a graph
  - Explore "deeper" in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex v that still has unexplored edges
  - When all of v's edges have been explored, **backtrack** to the vertex from which v was discovered

Depth-First Search

- Initialize
  - color all vertices white
- Visit each and every white vertex using DFS-Visit
- Each call to **DFS-Visit(*u*) roots a new tree** of the ***depth-first forest*** at vertex u
- A vertex is **white** if it is undiscovered
- A vertex is **gray** if it has been discovered but not all of its edges have been discovered
- A vertex is **black** after all of its adjacent vertices have been discovered (the adj. list was examined completely)

Depth-First Search

DFS(G)
1 **for** each vertex $u \in V[G]$
2 **do** $color[u] \leftarrow$ WHITE
3 $time \leftarrow 0$
4 **for** each vertex $u \in V[G]$
5 **do if** $color[u] =$ WHITE
6 **then** DFS-VISIT(u)

Init all vertices

DFS-VISIT(u)
1 $color[u] \leftarrow$ GRAY ▷ White vertex u discovered.
2 $d[u] \leftarrow time$ ▷ Mark with discovery time.
3 $time \leftarrow time + 1$ ▷ Tick global time.
4 **for** each $v \in Adj[u]$ ▷ Explore all edges (u, v).
5 **do if** $color[v] =$ WHITE
6 **then** DFS-VISIT(v)
7 $color[u] \leftarrow$ BLACK ▷ Blacken u; it is finished.
8 $f[u] \leftarrow time$ ▷ Mark with finishing time.
9 $time \leftarrow time + 1$ ▷ Tick global time.

Visit all children recursively

Depth-First Search: Algorithm

```
DFS(G)
{
   for each vertex u ∈ V[G]
   {
      color[u] = WHITE;
      p[u] = NIL;
   }
   time = 0;
   for each vertex u ∈ V[G]
   {
      if (color[u] == WHITE)
         DFS_Visit(u);
   }
}
```

```
DFS_Visit(u)
{
   color[u] = GRAY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]

      if (color[v] == WHITE)
         p[v] = u;
         DFS_Visit(v);

   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

Depth-First Search: Algorithm

```
DFS(G)
{
   for each vertex u ∈ V[G]
   {
      color[u] = WHITE;
      p[u] = NIL;
   }
   time = 0;
   for each vertex u ∈ V[G]
   {
      if (color[u] == WHITE)
         DFS_Visit(u);
   }
}
```

```
DFS_Visit(u)
{
   color[u] = GRAY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]

      if (color[v] == WHITE)
          p[v] = u;
          DFS_Visit(v);

   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

What does `d[u]` *represent?*

Depth-First Search: Algorithm

```
DFS(G)
{
   for each vertex u ∈ V[G]
   {
      color[u] = WHITE;
      p[u] = NIL
   }
   time = 0;
   for each vertex u ∈ V[G]
   {
      if (color[u] == WHITE)
         DFS_Visit(u);
   }
}
```

```
DFS_Visit(u)
{
   color[u] = GRAY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]

      if (color[v] == WHITE)
          p[v] = u;
          DFS_Visit(v);

   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

What does `f[u]` *represent?*

Depth-First Search: Algorithm

```
   DFS(G)
   {
1    for each vertex u ∈ V[G]
     {
2       color[u] = WHITE;
3       p[u] = NIL
     }
4    time = 0;
5    for each vertex u ∈ V[G]
     {
6       if (color[u] == WHITE)
7          DFS_Visit(u);
     }
   }
```

```
   DFS_Visit(u)
   {
1  color[u] = GRAY;
2  time = time+1;
3  d[u] = time;
4  for each v ∈ Adj[u]

5     if (color[v] == WHITE)
6        p[v] = u;
7        DFS_Visit(v);

8  color[u] = BLACK;
9  time = time+1;
10 f[u] = time;
   }
```

Will all vertices eventually be colored black?

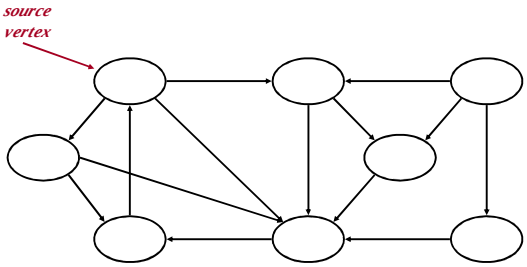
Depth-First Search: DFS Procedure

- Lines 1-3 paint all vertices white and initialize their p fields to NIL.
- Line 4 resets the global time counter.
- Lines 5-7 check each vertex in V in turn and, when a white vertex is found, visit it using DFS_Visit.
  - Every time DFS-Visit(u) is called in line 7, vertex u becomes the ***root of a new tree*** in the depth-first forest.
- When DFS returns, every vertex u has been assigned a ***discovery time*** d[u] and a ***finishing time*** f[u].

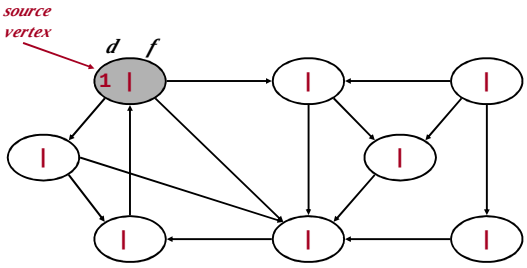
Depth-First Search: DFS-Visit Procedure

- In each call DFS-Visit(u), vertex u is initially white.
- Line 1 paints u gray.
- Line 2 increments the global variable time.
- Line 3 records the new value of time as the discovery time d[u].
- Lines 4-7 examine each vertex v adjacent to u and recursively visit v if it is white.
  - As each vertex v ∈ Adj[u] is considered in line 4, we say that edge (u, v) is ***explored*** by the depth-first search.
- Finally, after every edge leaving u has been explored, lines 8-10 paint u black and record the finishing time in f[u].

DFS Example



DFS Example



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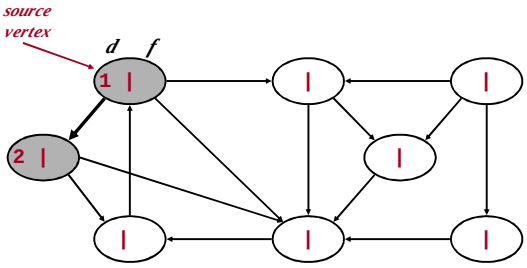
DFS Example



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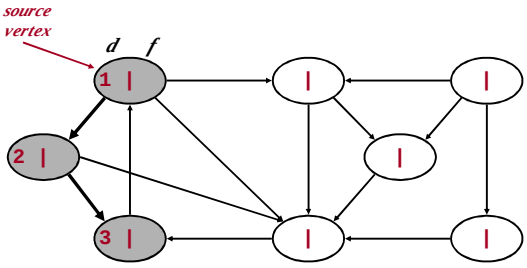
DFS Example



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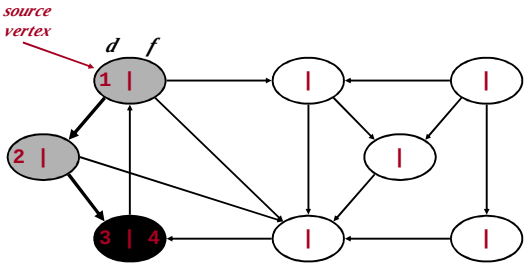
DFS Example



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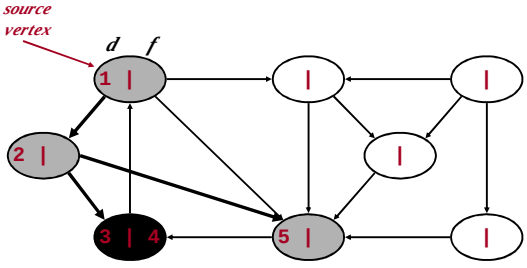
DFS Example



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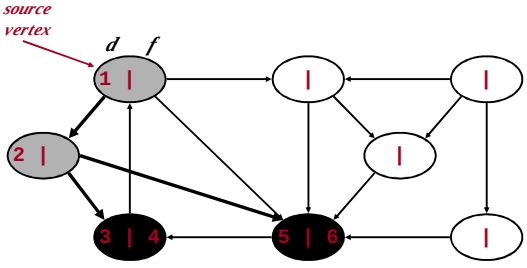
DFS Example



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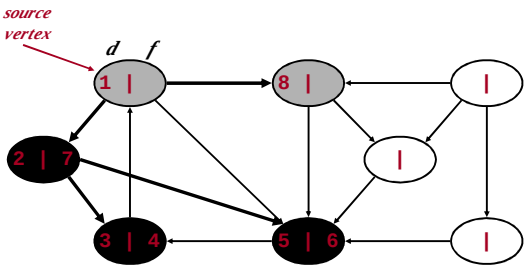
DFS Example



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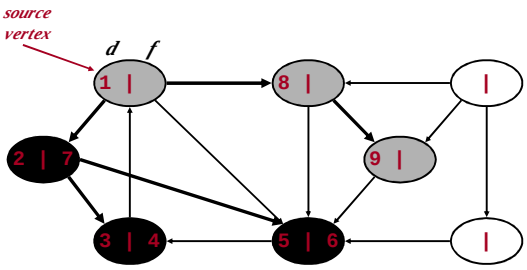
DFS Example



What is the structure of the gray vertices?
What do they represent?

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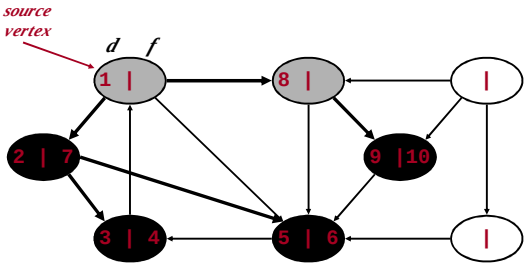
DFS Example



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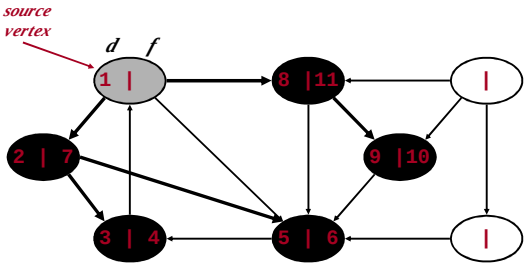
DFS Example



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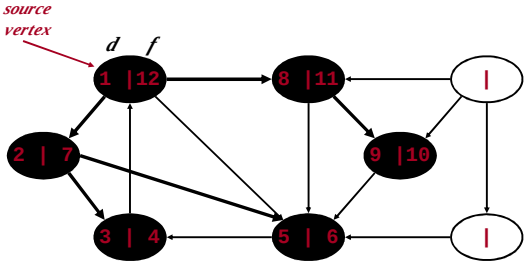
DFS Example



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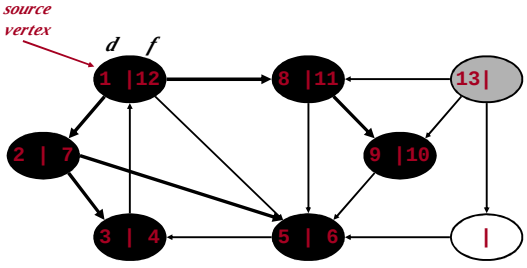
DFS Example



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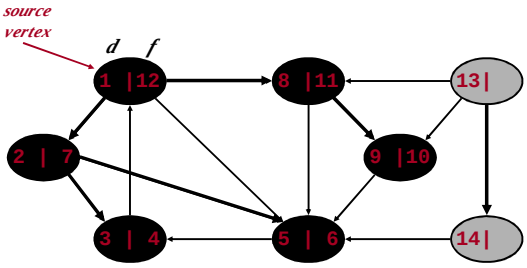
DFS Example



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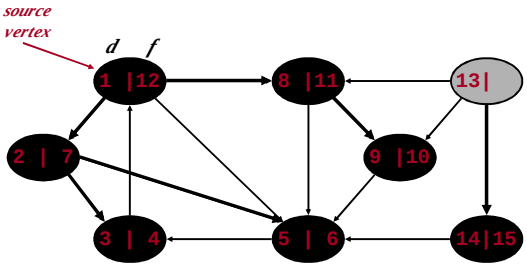
DFS Example



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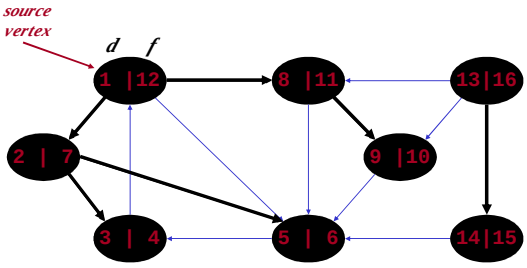
DFS Example



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Depth-First Search: Algorithm

```
DFS(G)
{
   for each vertex u ∈ V[G]
   {
      color[u] = WHITE;
      p[u] = NIL
   }
   time = 0;
   for each vertex u ∈ V[G]
   {
      if (color[u] == WHITE)
         DFS_Visit(u);
   }
}
```

```
DFS_Visit(u)
{
   color[u] = GRAY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]

      if (color[v] == WHITE)
         p[v] = u;
         DFS_Visit(v);

   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

What will be the running time?

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Depth-First Search: Algorithm

```
DFS(G)
{
   for each vertex u ∈ V[G]
   {
      color[u] = WHITE;
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   }
   time = 0;
   for each vertex u ∈ V[G]
   {
      if (color[u] == WHITE)
         DFS_Visit(u);
   }
}
```

Θ(*V*)

```
DFS_Visit(u)
{
   color[u] = GRAY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]

      if (color[v] == WHITE)
         p[v] = u;
         DFS_Visit(v);

   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

How many times will DFS_Visit() actually be called? Θ(*V*)
How many times will *if* statement be executed in all *O*(*V*) DFS_Visit calls? Θ(*E*)

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Depth-First Search: Algorithm

```
DFS(G)
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   for each vertex u ∈ V[G]
   {
      color[u] = WHITE;
      p[u] = NIL
   }
   time = 0;
   for each vertex u ∈ V[G]
   {
      if (color[u] == WHITE)
         DFS_Visit(u);
   }
}
```

```
DFS_Visit(u)
{
   color[u] = GRAY;
   time = time+1;
   d[u] = time;
   for each v ∈ Adj[u]

      if (color[v] == WHITE)
         p[v] = u;
         DFS_Visit(v);

   color[u] = BLACK;
   time = time+1;
   f[u] = time;
}
```

So, running time of DFS = Θ(V+E)

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DFS: Running Time

- Running time
  - the loops in **DFS** take time $\Theta(V)$ each, *excluding* the time to execute **DFS-Visit**
  - **DFS-Visit** is called once for every vertex
    - its only invoked on white vertices, and
    - paints the vertex gray immediately
  - for each **DFS-Visit(v)** a loop iterates over all **Adj[v]**
  - the *total cost* for **DFS-Visit** is $\Theta(E)$

$$\sum_{v \in V} |Adj[v]| = \Theta(E)$$

  - **the running time of DFS is $\Theta(V+E)$**

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Predecessor Subgraph

- Define slightly different from **BFS**
- Defined the *predecessor subgraph* of G as $G_p = (V, E_p)$, where
  - $E_p = \{(p[v], v) \in E : v \in V \text{ and } p[v] \neq \text{NIL}\}$
- The PD subgraph of a depth-first search forms a **depth-first forest** composed of several depth-first trees
- The edges in G_p are called *tree edges*

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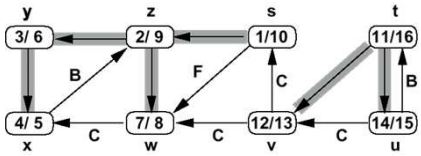
DFS Timestamping

- The **DFS** algorithm maintains a monotonically increasing global clock
  - discovery time $d[u]$ and finishing time $f[u]$
- For every vertex u, the inequality $d[u] < f[u]$ must hold



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DFS Timestamping

- Vertex u is
  - white before time $d[u]$
  - gray between time $d[u]$ and time $f[u]$, and
  - black thereafter
- Notice the structure througout the algorithm.
  - gray vertices form a ***linear chain***
  - correponds to a ***stack*** of vertices that have not been exhaustively explored (DFS-Visit started but not yet finished)
  - What is the ***size*** of the ***stack*** in DFS?

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DFS Parenthesis Theorem

- Discovery and finish times have parenthesis structure
  - represent *discovery* of u with *left* parenthesis "(u"
  - represent *finishing* of u with *right* parenthesis "u)"
  - history of discoveries and finishings makes a well-formed expression (parenthesis are properly nested)

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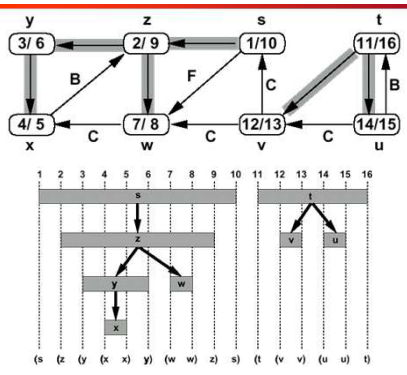
DFS Parenthesis Theorem



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DFS Parenthesis Theorem

- For any two distinct vertices u and v, exactly one of the following **three conditions** holds:
  - The intervals $[d[u], f[u]]$ and $[d[v], f[v]]$ are ***disjoint***
    - neither u nor v is a **descendant** (child) of the other in the depth-first forest
  - The interval $[d[u], f[u]]$ is contained entirely within the interval $[d[v], f[v]]$
    - $d[v] < d[u] < f[u] < f[v]$
    - u is a **descendant** of v in the depth-first forest
  - The interval $[d[v], f[v]]$ is contained entirely within the interval $[d[u], f[u]]$
    - $d[u] < d[v] < f[v] < f[u]$
    - v is a **descendant** of u in the depth-first forest

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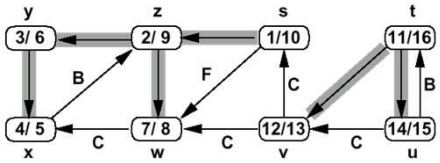
DFS Edge Classification

- **Tree** edge (**gray** to **white**)
  - encounter new vertices (white)
  - included in **depth-first forest** G_p
- **Back** edge (**gray** to **gray**)
  - from *descendant* to *ancestor*
  - non-tree edges in **depth-first tree**



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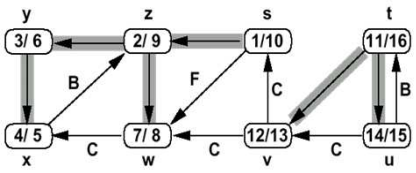
DFS Edge Classification

- **Forward** edge (**gray** to **black**): if $d[u] < d[v]$
  - from *ancestor* to *descendant*
  - non-tree edges in **depth-first tree**
- **Cross** edge (**gray** to **black**): if $d[u] > d[v]$
  - are all other edges
  - between trees or subtrees



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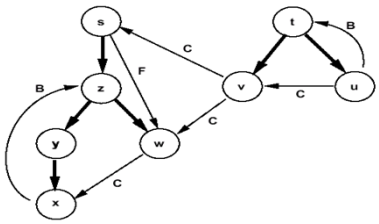
DFS Edge Classification

- **Tree** and **back** edges are important
- Most algorithms do not distinguish between **forward** and **cross** edges



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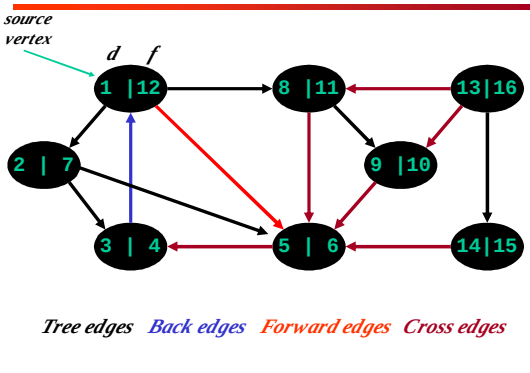
DFS Example



Tree edges *Back edges* *Forward edges* *Cross edges*

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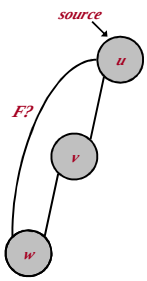
DFS: Kinds Of Edges

- **Theorem 22.10:**
  - If G is **undirected**, a DFS produces only **tree** and **back** edges
- Proof by contradiction:
  - Assume there's a **forward** edge
    - But *F?* edge must actually be a **back** edge (*why?*)
    - Since edge (w, u) is from *descendant* to *ancestor* (*gray to gray*)



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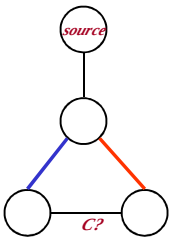
DFS: Kinds Of Edges

- **Theorem 22.10:**
  - If G is **undirected**, a DFS produces only **tree** and **back** edges
- Proof by contradiction:
  - Assume there's a **cross** edge
    - But ***C*?** edge **cannot** be **cross**:
    - must be explored from one of the vertices it connects, becoming a **tree** vertex, before other vertex is explored
    - So in fact the picture is wrong…**both** blue and red edges **cannot** in fact be **tree** edges.
    - One should be **tree** edge and
    - the other should be **back** edge (cycle graph)



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DFS and Graph Cycles

- Theorem:
- An **undirected** graph is ***acyclic*** (no cycles) iff a DFS yields **no back** edges
  - If acyclic, no back edges (because a back edge implies a cycle
  - If no back edges, acyclic
    - No back edges implies only tree edges (*Why?*)
    - Only ***tree edges*** implies we have a ***tree or a forest***
    - Which by definition is acyclic
- Thus, can run DFS to find whether a graph has a cycle. (*How*)

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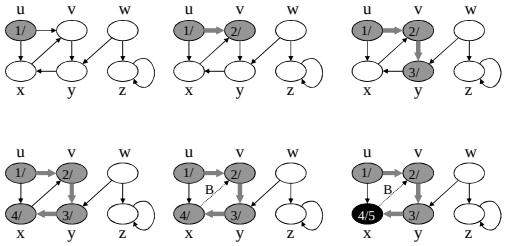
DFS Example



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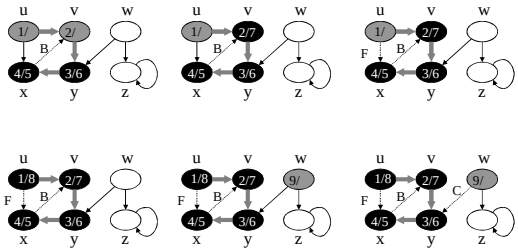
DFS Example



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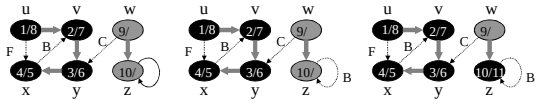
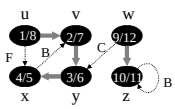
DFS Example





If you have a back edge in a directed graph, then you may have a cycle or must have a cycle?

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