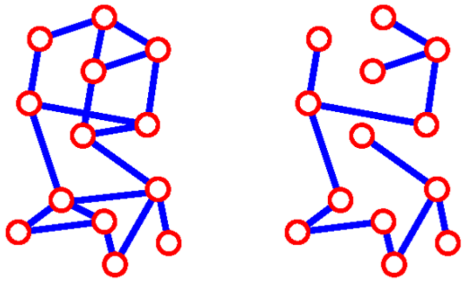
Minimum Spanning Trees

Outline: MST

- Minimum Spanning Tree
- Generic MST Algorithm
- Kruskal’s Algorithm (Edge Based)
- Prim’s Algorithm (Vertex Based)

Spanning Tree

- A **spanning tree** of *G* is a subgraph which
  - is tree (acyclic)
  - and connect all the vertices in *V*.



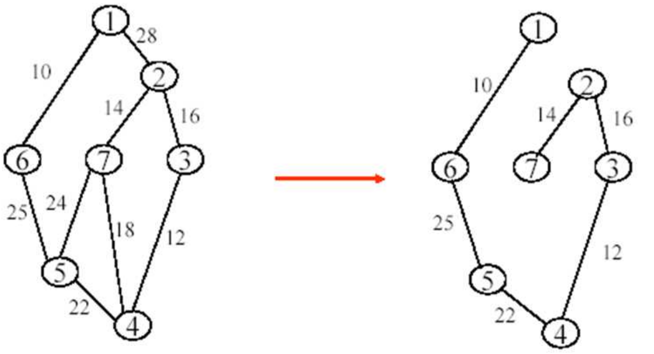
G spanning tree of **G**

  - Spanning tree has only $|V| - 1$ edges.

Minimum Spanning Tree

- **Input:**
  - Undirected connected graph $G = (V, E)$ and weight function $w : E \rightarrow \mathbf{R}$,
- **Output:**
  - A Minimum spanning tree *T*: tree that connects all the vertices and minimizes $w(T) = \sum_{(u,v)\in T} w(u,v)$
- Greedy Algorithms
  - Generic MST algorithm
  - Kruskal's algorithm
  - Prim's algorithm

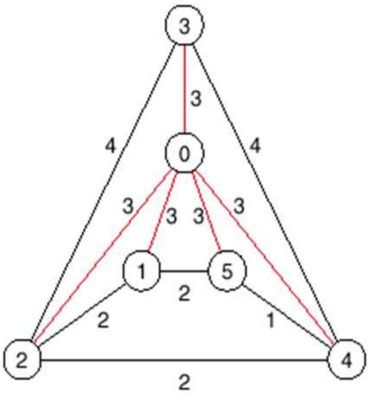
Example: MST



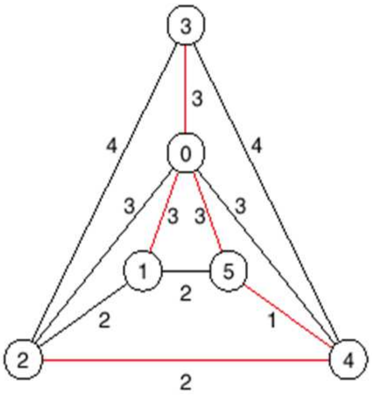
G

Minimum Cost Spanning Tree

Example: MST



Weight sum of tree = 15



Weight sum of tree = 12

Generic MST Algorithm

- Our goal is to build a spanning tree by adding one edge at a time to a **set** ***A*** in a “***greedy***” fashion.
- Basically, we just need to somehow guarantee ourselves that at each step, the current set can be “extended” to an MST.
- Strategy:
  - Grow the MST ***one edge at a time***, ensuring that the partial solution remains a subset of some MST
- How do we do that?

Generic MST Algorithm

- Let’s assume that our current set of edges ***A*** already satisfies the property that ***A*** can be extended to an MST.
- Question:
  - What edges can we add to ***A*** to maintain the property?
- Answer:
  - an edge ***e*** such that ***w(e) ≤ w(other edges)*** and
  - $A \cup \{e\}$ is ***acyclic***
- We will call such edge a ***safe edge*** if it also doesn’t create a cycle when added to ***A***.

Generic MST Algorithm

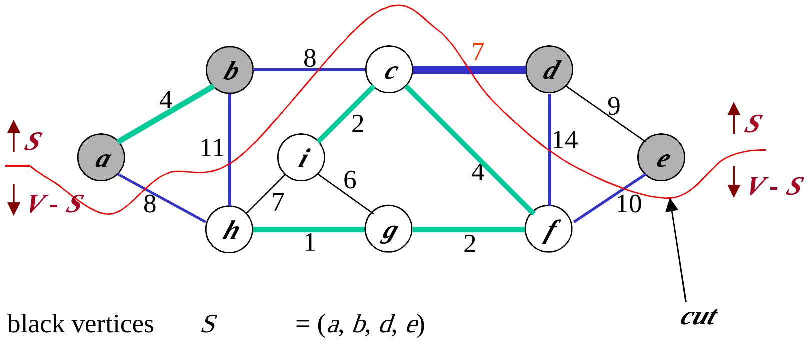
```
Generic-MST(G, w)
1. A = { }
2. while A does not form a spanning tree
3.     find an edge (u, v) that is safe for A
4.         Add (u, v) to A
5. return A
```

- How to find a **safe** edge?

Generic MST Algorithm

- Definitions:
  - A ***cut*** $(S, V - S)$ of $G = (V, E)$ is a partition of V into 2 sets
  - An edge $(u, v) \in E$ ***crosses*** the cut $(S, V - S)$ if one point is in S while the other point in $V - S$
  - A ***cut respects a set A of edges*** if **no edges** in A ***crosses*** the cut.
  - An edge is ***light*** if its weight is **minimum** of all edges crossing the cut

Generic MST Algorithm



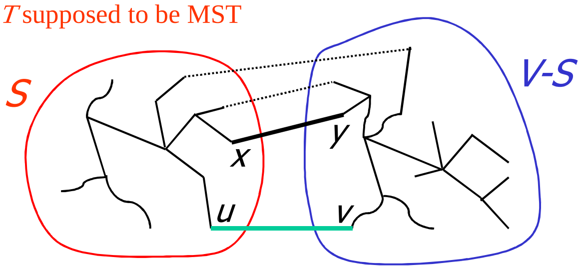
black vertices	S = (a, b, d, e)
white vertices	V-S = (c, i, h, g, f)
red line	*cut*
blue	edges crossing the cut
(*c*, *d*)	only light edge
green	a subset ***A*** of edges; which is a cut (S, V-S) **respects** to ***A***, since no edge of ***A*** crosses the cut.

Theorem 23.1 (CLRS)

- Let $G = (V, E)$ be a *connected, undirected* graph
- w is weight function $w: E \rightarrow \mathbf{R}$
- Let $A \subseteq E$ be included in some MST T for G
- Let (S, V - S) be any cut of G that respect A
- Let (u, v) be a light edge (**min-weight**) crossing the cut (S, V - S)
- Then, edge (u, v) is safe for A
    - i.e., $(u, v) \in T$, a **MST** of G

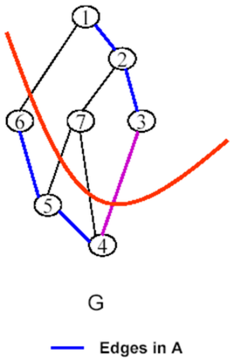
Theorem 23.1 (CLRS)

- Proof
  - suppose $(u, v) \notin T$
  - look at path from u to v in T
  - swap (x, y) with (u, v)
    - ➢ the first edge on path from u to v in T that crosses from S to $V - S$
  - this increases $w(T)$ – contradiction
    - ➢ T supposed to be MST



Generic MST Algorithm

- **Corollary 23.2 (CLRS):**
  - Let $A \subseteq E$ be included in some MST
  - Let $C_1 = (V_{C1}, E_{C1})$ and
  - $C_2 = (V_{C2}, E_{C2})$ be two distinct connected components (trees) in the forest $G_A = (V, A)$.
  - If (u, v) is a *light* edge crossing the cut = (V_{C1}, V_{C2}) then (u, v) is *safe* for A.



Generic MST Algorithm

```
Generic-MST($G$, $w$)
1.  $A$ = { }
2.  while $A$ does not form a spanning tree
3.          find an edge ($u$, $v$) that is safe for $A$
4.                  Add ($u$, $v$) to $A$
5.  return $A$
```

- As the algorithm proceeds:
  - A is always acyclic
- At any point of the execution of the algorithm,
  - Graph $G_A = (V, A)$ is a forest, and
  - Each ***connected component*** is a tree
- Also, any safe edge (u, v) for A
  - Connects distinct components of G_A,
  - Since $A \cup (u, v)$ must be acyclic

Generic MST Algorithm

```
Generic-MST($G$, $w$)
1.  $A$ = { }
2.  while $A$ does not form a spanning tree
3.          find an edge ($u$, $v$) that is safe for $A$
4.                  Add ($u$, $v$) to $A$
5.  return $A$
```

- The loop in lines 2-4 is executed $|V|$-1 times
- Initially when $A = \{\}$, there are $|V|$ trees in G_A
  - Each tree has only one vertex
- When G_A (forest) contains only a single tree, the algorithm terminates

Kruskal's Algorithm

- **Edge based** algorithm
- Greedy strategy:
  - From the remaining edges, select a ***least-cost*** edge that ***does not result in a cycle*** when added to the set of already selected edges
  - Repeat $|V|$-1 times

Kruskal's Algorithm

- INPUT:
  - edge-weighted graph $G = (V, E)$, with $|V| = n$
- OUTPUT:
  - a spanning tree A of G
    - touches all vertices,
    - has n-1 edges
    - of minimum cost (= total edge weight)
- Algorithm:
  - Start with A empty,
  - Add the edges one at a time, in increasing weight order
  - An edge is accepted, if it connects vertices of distinct trees (if the edge does not form a cycle in A)
  - until A contains n-1 edges

Kruskal's Algorithm

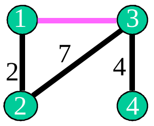
```
MST-Kruskal($G$, $w$)
1 $A \leftarrow \emptyset$
2 for each vertex $v \in V[G]$ do
3    Make-Set($v$)
4 sort the edges of $E$ by nondecreasing weight $w$
5 for each $(u, v) \in E$, in nondecreasing of weight do
6   if Find-Set($u$) $\neq$ Find-Set($v$) then
7      $A \leftarrow A \cup \{(u, v)\}$
8      Union(Set($u$),Set($v$))
9 return $A$
```

Kruskal's Algorithm

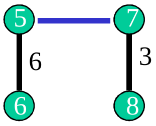
- **Lines 1-3 initialize the set *A* to empty set and create |V| trees, one containing each vertex.**
- **The edges in *E* are sorted into nondecreasing order by weight in line 4.**
- **The for loop in lines 5-8 checks, for each (u, v), whether the endpoints u and v belong to the same tree.**
    - **If they do, then the edge (u, v) cannot be added to the forest without creating a cycle, and the edge is discarded.**
    - **Otherwise, the two vertices belong to different trees.**
    - **In this case, the edge (u, v) is added to A in line 7, and the vertices in the two trees are merged in line 8.**

Data Structures For Kruskal’s Algorithm

- Does the addition of an edge (u, v) to A result in a cycle?
- Each component of A is a tree.
  - When u and v are in the
    - **same** component, the addition of the edge (u, v) creates a **cycle**.

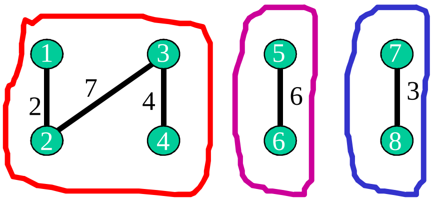


    - **different** components, the addition of the edge (u, v) **does not** create a cycle.



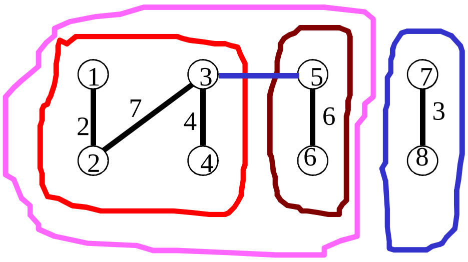
Data Structures For Kruskal’s Algorithm

- Each component of A is defined by the vertices in the component.
- Represent each component as a set of vertices.
  - {1, 2, 3, 4}, {5, 6}, {7, 8}
- Two vertices are in the same component iff they are in the same set of vertices.



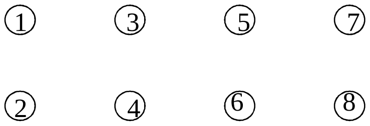
Data Structures For Kruskal’s Algorithm

- When an edge (u, v) is added to A, the two components that have vertices u and v combine to become a single component
- In our set representation of components, the set that has vertex u and the set that has vertex v are united.
  - {1, 2, **3**, 4} + {**5**, 6} ➔ {1, 2, 3, 4, 5, 6}



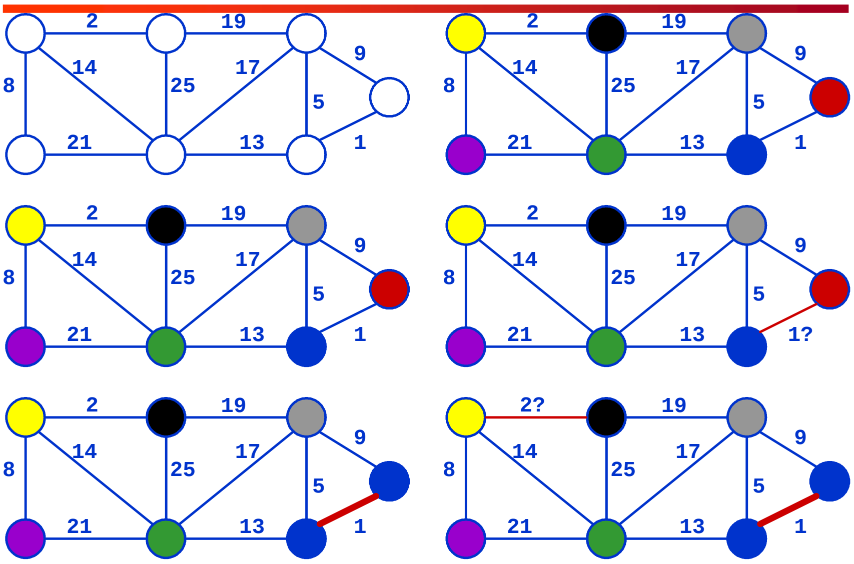
Data Structures For Kruskal’s Algorithm

- Initially, A is empty

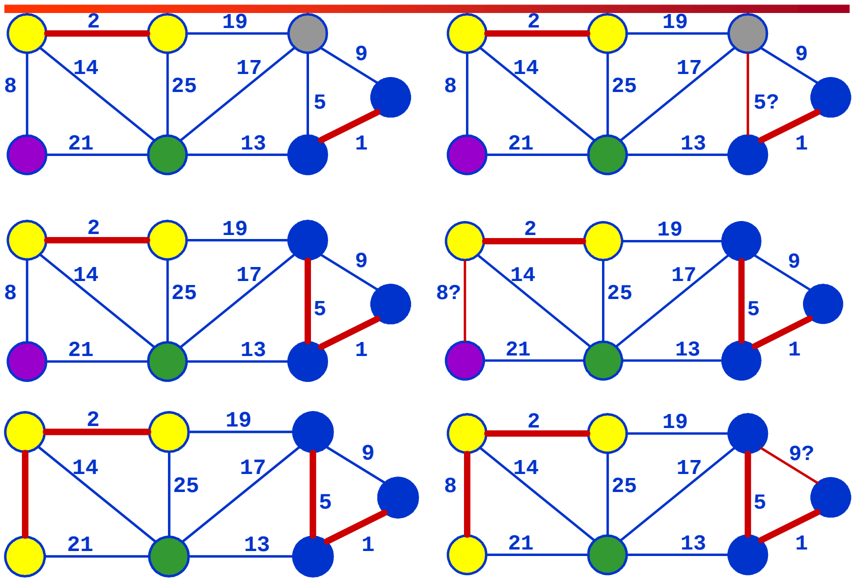


- Initial sets are:
  - {1} {2} {3} {4} {5} {6} {7} {8}
- Does the addition of an edge (u, v) to A result in a cycle? If not, add edge to A
  - s_1 = Find-Set(u); s_2 = Find-Set(v);
  - if ($s_1 \neq s_2$) then Union(s_1, s_2);

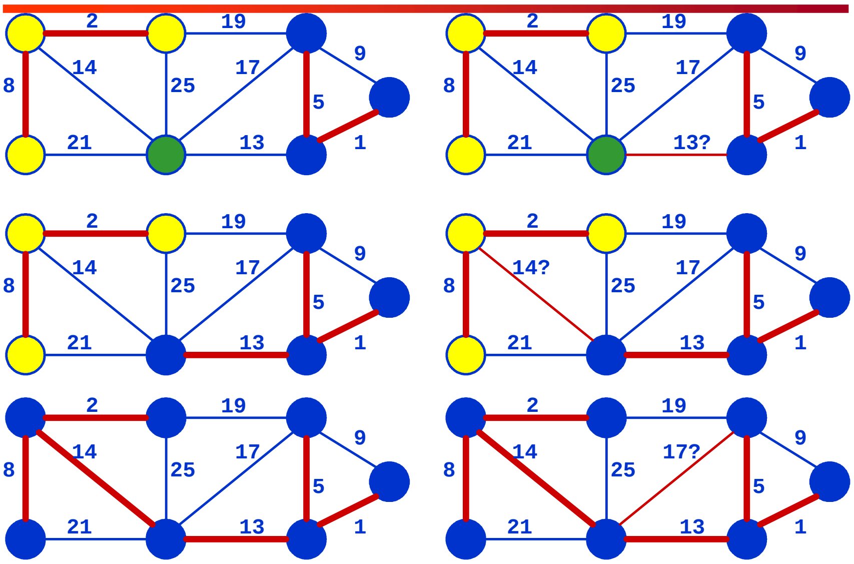
Kruskal's Algorithm



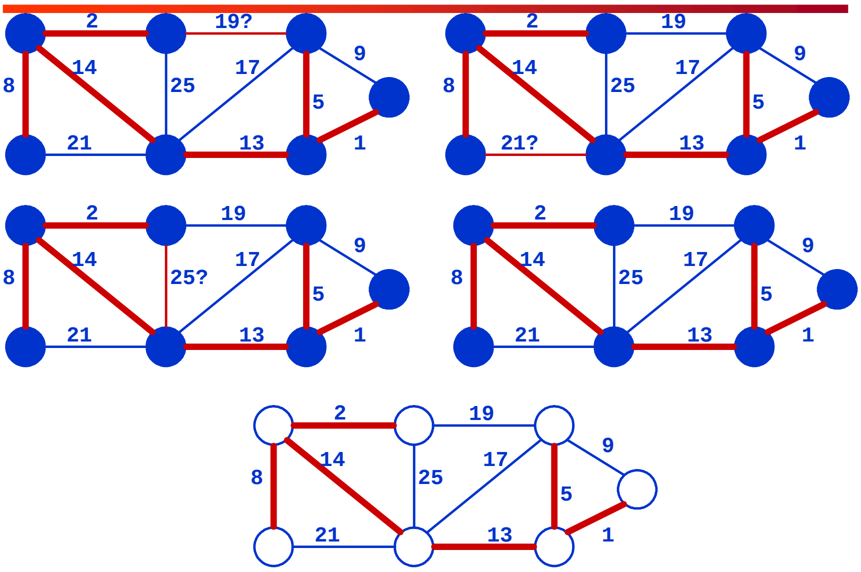
Kruskal's Algorithm



Kruskal’s Algorithm



Kruskal’s Algorithm



Kruskal's Algorithm

```
MST-Kruskal(G, w)
1 A ← ∅
2 for each vertex v ∈ V[G] do
3    Make-Set(v)
4 sort the edges of E by nondecreasing weight w
5 for each (u, v)∈E, in nondecreasing of weight do
6   if Find-Set(u) ≠ Find-Set(v) then
7      A ← A ∪ {(u, v)}
8      Union(Set(u),Set(v))
9 return A
```

Running Time of Kruskal's Algorithm

- Kruskal's Algorithm consists of two stages.
  - Initializing the set A in line 1 takes $O(1)$ time.
  - Sorting the edges by weight in line 4.
    - takes $O(E \lg E)$
  - Performing
    - $|V|$ MakeSet() operations   for loop in lines 2-3.
    - $|E|$ FindSet(),   for loop in lines 5-8.
    - $|V|$ - 1 Union(),   for loop in lines 5-8.
    - which takes $O(V + E)$
- The total running time is
  - $O(E \lg E)$
  - Observing that $|E| < |V|^2$ we have $\lg |E| = O(\lg V)$,
    So total running time becomes $O(E \lg V)$.

Prim’s Algorithm

- Prim’s algorithm has the property that edges in the set A always form a single tree.
- The tree starts from an arbitrary root vertex r and grows until the tree spans all the vertices in V.
- At each step, a light edge is added to the tree A that connects A to an isolated vertex of $G_A = (V, A)$.
  - Adds only edges that are safe for A.
  - When algorithm terminates, edges in A form MST.
  - MST A for G: $A=\{(v, p[v]): v \varepsilon V-\{r\}\}$.
- **Vertex based** algorithm.
- Grows one tree, **one vertex at a time**

Prim’s Algorithm

```
MST-Prim(G,w,r) //G: graph with weight w and a root vertex r
for each u ∊ V[G]
  key[u] ← ∞
  p[u] ← NIL
key[r] ← 0
Q = BuildMinHeap(V,key); // Q – vertices out of T
while Q ≠ ∅ do
  u ← ExtractMin(Q)  // making u part of T
     for each v ∊ Adj[u] do
        if v ∊ Q and w(u,v) < key[v] then
            p[v]      ← u
            key[v]    ← w(u,v)
```

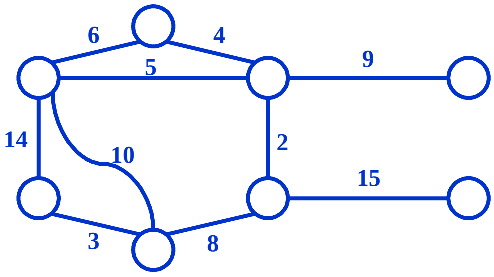
updating keys

- All vertices that are not in tree reside in min-priority queue Q based on a key field.
- For each vertex v, key[v] is min weight of any edge connecting v to a vertex in tree.
- key[v]= ∞ if there is no edge and p[v] names parent of v in tree.
- When algorithm terminates the min-priority queue Q is empty.

Prim's Algorithm

- Lines 1-5 set the key of each vertex to ∞ (except root r, whose key is set to 0 first vertex processed). Also, set parent of each vertex to NIL, and initialize min-priority queue Q to contain all vertices.
- Line 7 identifies a vertex $u \in Q$ incident on a light edge crossing cut (V-Q, Q) (except first iteration, u=r due to line 4).
- Removing u from set Q adds it to set Q- V of vertices in tree, thus adding (u, p[u]) to A.
- The for loop of lines 8-11 update *key* and p fields of every vertex v adjacent to u but not in tree.

Run on example graph



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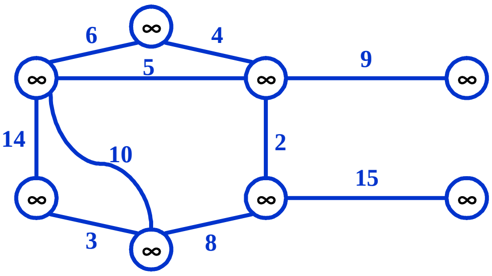
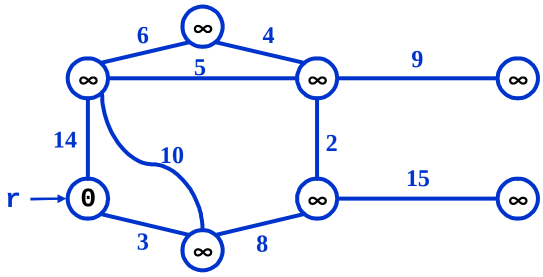


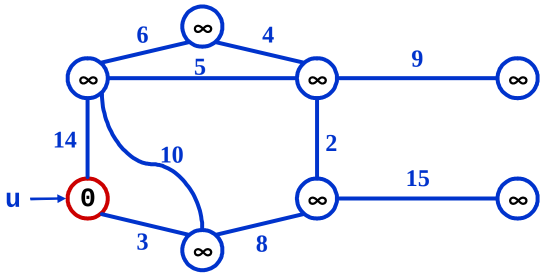
key[*u*] = ∞

Run on example graph



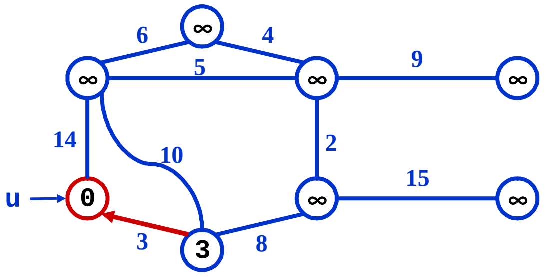
Pick a start vertex r

Run on example graph



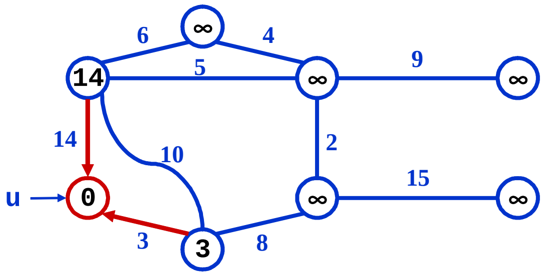
Red vertices have been removed from Q

Run on example graph



Red arrows indicate parent pointers

Run on example graph



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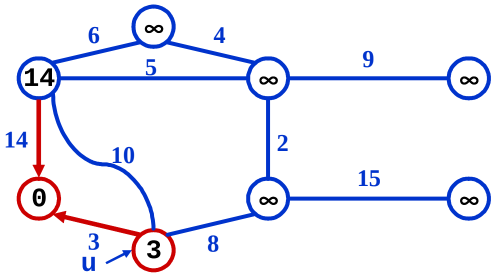

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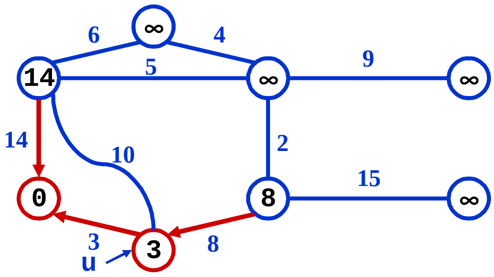

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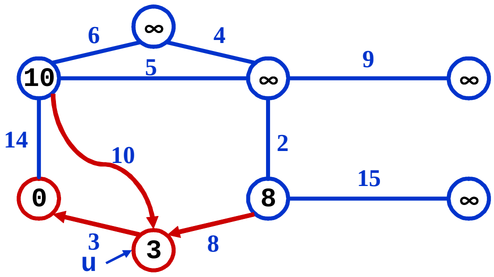

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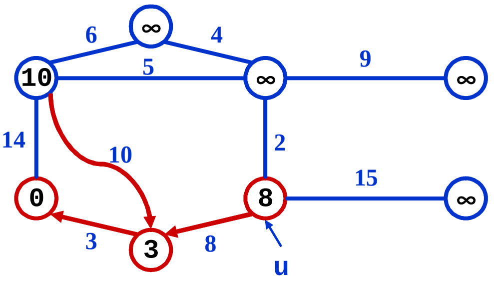

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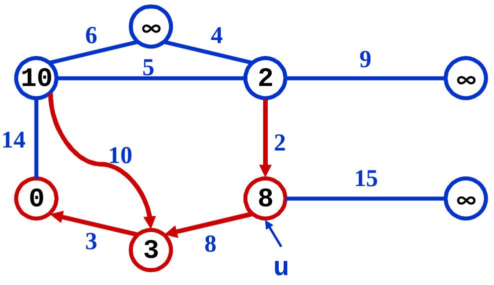

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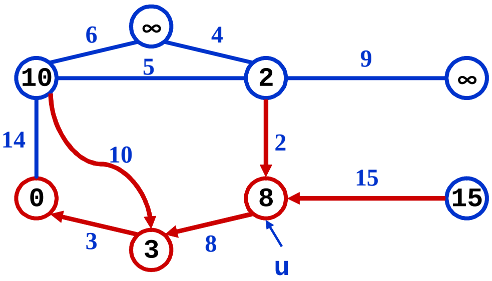

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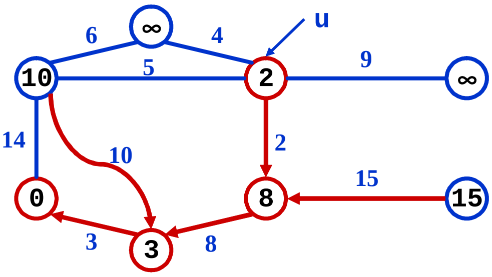

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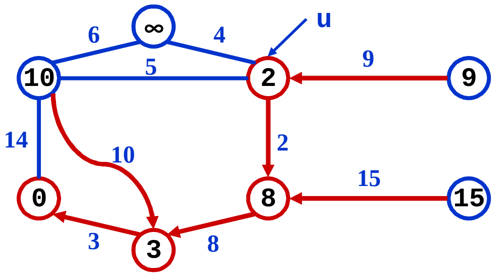

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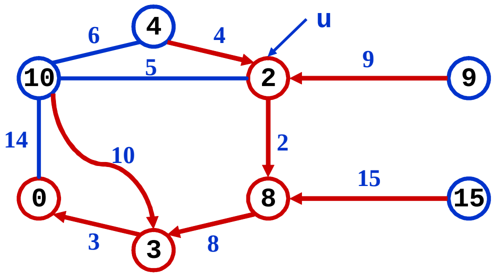

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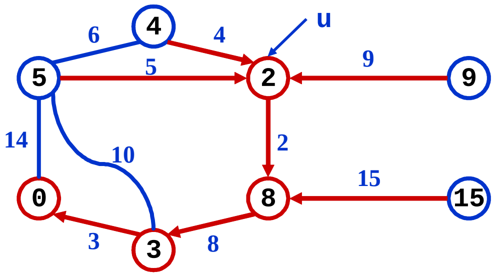

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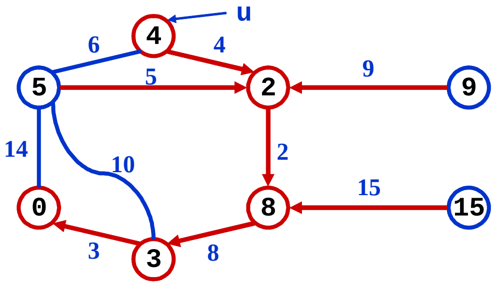

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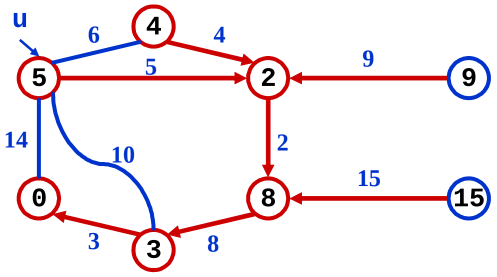

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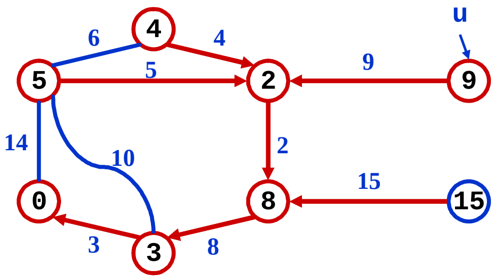

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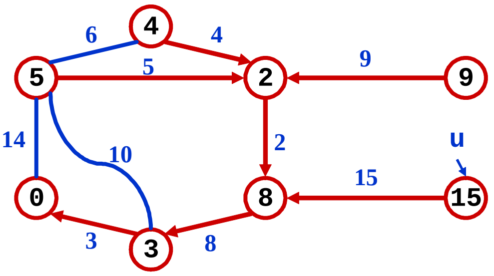


Prim's Running Time

- *What is the hidden cost in this code?*

Extract-Min is executed |*V*| times

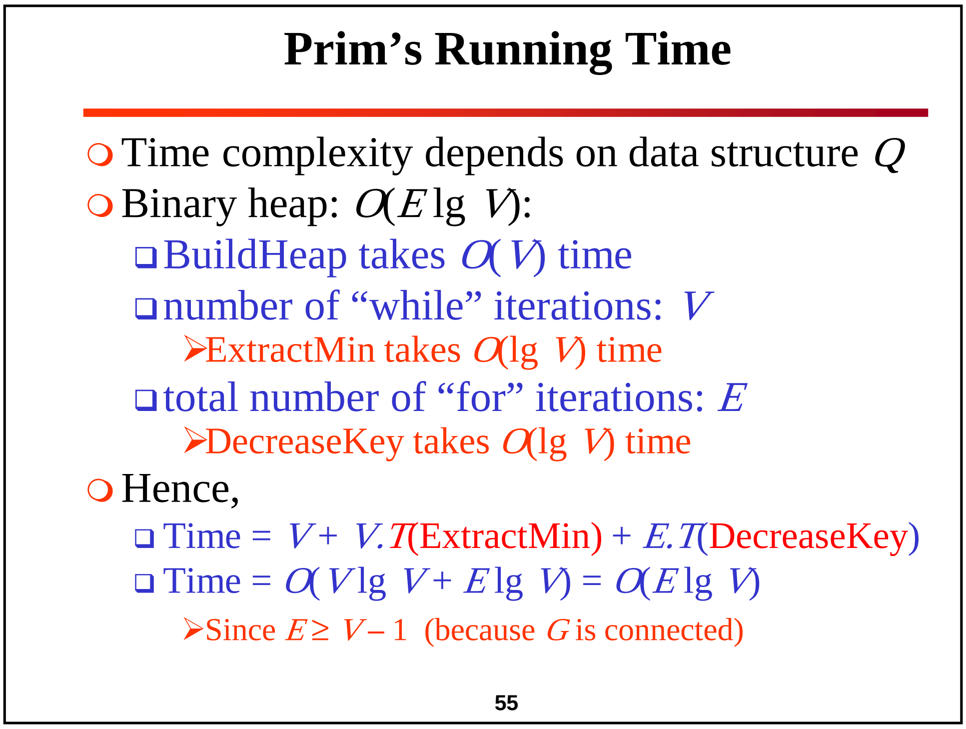
Decrease-Key is executed *O*(|*E*|) times

while loop is executed |*V*| times

```
MST-Prim(G,w,r)
for each u ∈ V[Q]
  key[u] ← ∞
  p[u] ← NIL
key[r] ← 0
Q = BuildHeap(V,key); //Q – vertices out of T
while Q ≠ ∅ do
  u ← ExtractMin(Q)   // making u part of T
     for each v ∈ Adj[u] do
          if v ∈ Q and w(u,v) < key[v] then
               p[v]     ← u
               key[v]   ← w(u,v)
```

updating keys

`DecreaseKey(v, w(u,v));`



Prim’s Running Time

- Time complexity depends on data structure Q
- Binary heap: $O(E \lg V)$:
  - BuildHeap takes $O(V)$ time
  - number of “while” iterations: V
    - ExtractMin takes $O(\lg V)$ time
  - total number of “for” iterations: E
    - DecreaseKey takes $O(\lg V)$ time
- Hence,
  - Time = $V + V.T(\text{ExtractMin}) + E.T(\text{DecreaseKey})$
  - Time = $O(V \lg V + E \lg V) = O(E \lg V)$
    - Since $E \geq V - 1$ (because G is connected)