Chapter 2
Getting Started

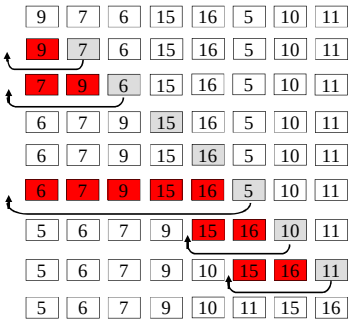
1

Outlines: Getting Started

- Sequential Approach
  - Insertion Sort
- Divide-and-Conquer Approach
  - Merge Sort

2

Insertion Sort Execution Example



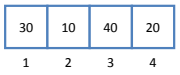
3

An Example: Insertion Sort

InsertionSort(A, n) {
  for i = 2 to n {
    key = A[i] — next key
    j = i - 1; — go left
    while (j > 0) and (A[j] > key) { — find place for key
      A[j+1] = A[j] — shift sorted right
      j = j – 1 — go left
    }
    A[j+1] = key — put key in place
  }
}

4

An Example: Insertion Sort

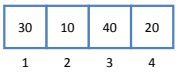


i = ∅ j = ∅ key = ∅
A[j] = ∅ A[j+1] = ∅

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

5

An Example: Insertion Sort

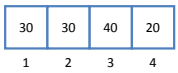


i = 2 j = 1 key = 10
A[j] = 30 A[j+1] = 10

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

6

An Example: Insertion Sort

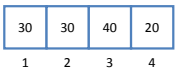


i = 2 j = 1 key = 10
A[j] = 30 A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

7

An Example: Insertion Sort

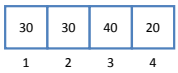


i = 2 j = 1 key = 10
A[j] = 30 A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

8

An Example: Insertion Sort

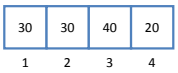


i = 2 j = 0 key = 10
A[j] = ∅ A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

9

An Example: Insertion Sort



i = 2 j = 0 key = 10
A[j] = ∅ A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
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            j = j - 1
      }
      A[j+1] = key
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}
```

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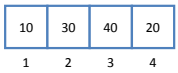
An Example: Insertion Sort



i = 2 j = 0 key = 10
A[j] = ∅ A[j+1] = 10

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
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            j = j - 1
      }
      A[j+1] = key
  }
}
```

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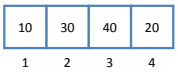
An Example: Insertion Sort



i = 3 j = 0 key = 10
A[j] = ∅ A[j+1] = 10

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

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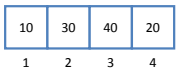
An Example: Insertion Sort



i = 3 j = 0 key = 40
A[j] = ∅ A[j+1] = 10

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
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```

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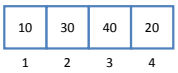
An Example: Insertion Sort



i = 3 j = 0 key = 40
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InsertionSort(A, n) {
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      }
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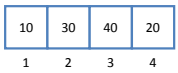
An Example: Insertion Sort



i = 3 j = 2 key = 40
A[j] = 30 A[j+1] = 40

```
InsertionSort(A, n) {
  for i = 2 to n {
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            j = j - 1
      }
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```
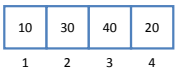

15

An Example: Insertion Sort



i = 3 j = 2 key = 40
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```
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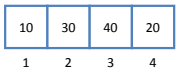
An Example: Insertion Sort



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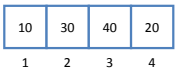
An Example: Insertion Sort



i = 4 j = 2 key = 40
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```
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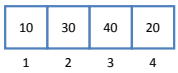

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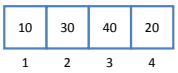

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20

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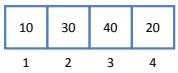

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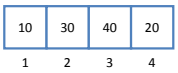

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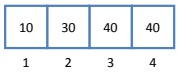
An Example: Insertion Sort



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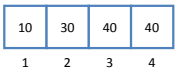
23

An Example: Insertion Sort



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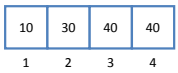
24

An Example: Insertion Sort



i = 4 j = 3 key = 20
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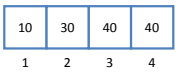
```

25

An Example: Insertion Sort



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            j = j - 1
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      A[j+1] = key
  }
}
```

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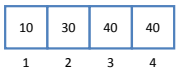
An Example: Insertion Sort



i = 4 j = 2 key = 20
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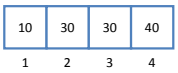
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

27

An Example: Insertion Sort



i = 4 j = 2 key = 20
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```
InsertionSort(A, n) {
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      while (j > 0) and (A[j] > key) {
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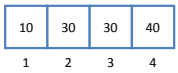
```

28

An Example: Insertion Sort



i = 4 j = 2 key = 20
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```
InsertionSort(A, n) {
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```

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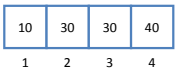
An Example: Insertion Sort



i = 4 j = 1 key = 20
A[j] = 10 A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

30

An Example: Insertion Sort

10	30	30	40
1	2	3	4

i = 4 j = 1 key = 20
A[j] = 10 A[j+1] = 30

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
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An Example: Insertion Sort

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1	2	3	4

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      }
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An Example: Insertion Sort

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1	2	3	4

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```
InsertionSort(A, n) {
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      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

Done!

Insertion Sort

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

How many times will this while loop execute?

Insertion Sort

Statement	Effort
`InsertionSort(A, n) {`	
`  for i = 2 to n {`	$c_1 n$
`      key = A[i]`	$c_2(n-1)$
`      j = i - 1`	$c_3(n-1)$
`      while (j > 0) and (A[j] > key) {`	$c_4 C$
`            A[j+1] = A[j]`	$c_5(C-(n-1))$
`            j = j - 1}`	$c_6(C-(n-1))$
`      A[j+1] = key`	$c_7(n-1)$
`  }}`	

$C = t_2 + t_3 + \ldots + t_n$ where t_i is number of **while** expression evaluations for the i^{th} **for** loop iteration

Body of the **while** statement is executed $= (t_2 - 1) + (t_3 - 1) + \ldots + (t_n - 1)$
$= t_2 + t_3 + \ldots + t_n - (n-1) = C - (n-1)$

Analyzing Insertion Sort

$T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 C + c_5(C - (n-1)) + c_6(C - (n-1)) + c_7(n-1)$

$= c_8 C + c_9 n + c_{10}$

- What can C be?
  - Best case -- inner loop body never executed (array is sorted)
    - $t_i = 1 \rightarrow T(n)$ is a linear function $= a.n + b$
  - Worst case -- inner loop body executed for all previous elements (array sorted in reverse order)
    - $t_i = i \rightarrow T(n)$ is a quadratic function $= a.n^2 + b.n + c$
  - *If T is a quadratic function, which terms in the above equation matter?*

$c_1 n$
$c_2(n-1)$
$c_3(n-1)$
$c_4 C$
$c_5(C-(n-1))$
$c_6(C-(n-1))$
$c_7(n-1)$

Analyzing Insertion Sort (Cont.)

- Best Case
  - If A is sorted: $O(n)$ comparisons
- Worst Case
  - If A is reversed sorted: $O(n^2)$ comparisons
- Average Case
  - If A is randomly sorted: $O(n^2)$ comparisons

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Divide-and-Conquer

- Recursive in structure
  - *Divide* the problem into several smaller sub-problems that are similar to the original but smaller in size
  - *Conquer* the sub-problems by solving them recursively. If they are small enough, just solve them in a straightforward manner.
  - *Combine* the solutions to create a solution to the original problem

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An Example: Merge Sort

- *Divide*: Divide the n-element sequence to be sorted into two subsequences of $n/2$ elements each
- *Conquer:* Sort the two subsequences recursively using merge sort.
- *Combine:* Merge the two sorted subsequences to produce the sorted answer.

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Merge Sort

```
MergeSort(A, left, right) {
  if (left < right) {
      mid = floor((left + right) / 2);
      MergeSort(A, left, mid);
      MergeSort(A, mid+1, right);
      Merge(A, left, mid, right);
  }
}

// Merge() takes two sorted subarrays of A and
// merges them into a single sorted subarray of A.
// It requires O(n) time
```

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Merge

```
Merge (A, p, q, r)
n1 = q – p + 1
n2 = r – q
Create arrays L[1 .. n1 + 1] and R[1 .. n2 + 1]
for i = 1 to n1
       L[i] = A[p + i – 1]
for j = 1 to n2
       R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r
       if (L[i] ≤ R[j]) {
              A[k] = L[i]
              i = i + 1 }
       else {
              A[k] = R[j]
              j = j + 1 }
```

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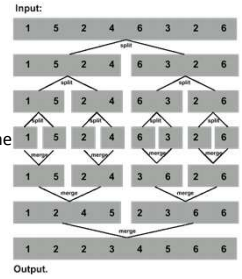
Merge Sort Revisited

- To sort n numbers
  - if $n = 1$ done!
  - recursively sort 2 lists of numbers $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ elements
  - merge 2 sorted lists in $O(n)$ time
- Strategy
  - break problem into similar (smaller) subproblems
  - recursively solve subproblems
  - combine solutions to answer



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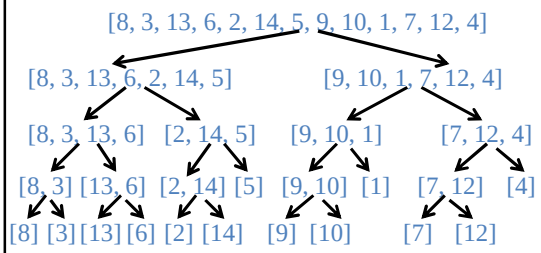
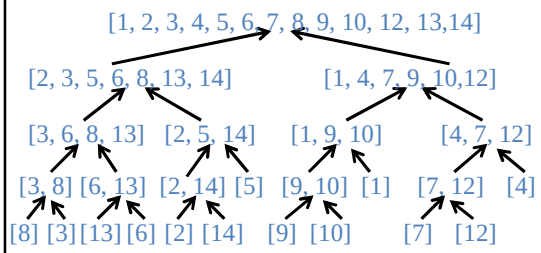
Merge Sort



Merge Sort



Analysis of Merge Sort

Statement	Effort
`MergeSort(A, left, right) {`	`T(n)`
`  if (left < right) {`	`Θ(1)`
`  mid = floor((left + right) / 2);`	`Θ(1)`
`  MergeSort(A, left, mid);`	`T(n/2)`
`  MergeSort(A, mid+1, right);`	`T(n/2)`
`  Merge(A, left, mid, right);`	`Θ(n)`
`  }`	
`}`	

- So T(n) = Θ(1) when n = 1, and
  2T(n/2) + Θ(n) + Θ(1) when n > 1
- Solving this recurrence (*how?*) gives **T(n) = n log n**
- This expression is a *recurrence*

Review: Analysis of Merge Sort

- Divide: computing the middle takes ***O*(1)**
- Conquer: solving 2 sub-problem takes **2*T*(*n*/2)**
- Combine: merging *n*-element takes ***O*(*n*)**
- Total:
  ***T*(*n*) = *O*(1)** if ***n* = 1**
  ***T*(*n*) = 2*T*(*n*/2) + *O*(*n*) + *O*(1)** if ***n* > 1**

  ⇒***T*(*n*) = *O*(*n* lg *n*)**
- Solving this recurrence (*how?*) gives *T*(*n*) = *O*(*n* lg *n*)
- This expression is a *recurrence*
- **T(n)** denote the number of operations required by an algorithm to solve a given class of problems.

Analysis of Merge Sort

Assume $n=2^k$ for $k>=1$

$T(n) = 2\,T(n/2) + bn + c$

$T(n/2) = 2T((n/2)/2) + b(n/2) + c$

$= 2[2T(n/4) + b(n/2) + c] + bn + c$

$= 4\,T(n/4) + bn + 2c + bn + c$

$= 4\,T(n/4) + 2bn + (1+2)\,c = 2^2\,T(n/2^2) + 2bn + (2^0+2^1)c$

$= 4\,[2T((n/4)/2) + b(n/4) + c] + 2bn + (1+2)c$

$= 8\,T(n/8) + 3bn + (1+2+4)c$

$= 2^3\,T(n/2^3) + 3bn + (2^0+2^1+2^2)c$

⋮

$= 2^k\,T(n/2^k) + kbn + (2^0+2^1+\ldots+2^{k-1})c$

$T(1) = a$, since $n=2^k$ → $\log n = (\log 2^k) = k$

$T(n) = 2^k.\ a + k\,bn + (2^0+2^1+\ldots+2^{k-1})\,c$

$= n\,.\,a + \log n\; b\,n + c.\quad \left(\sum_{i=0}^{k-1} 2^i = 2^{(k-1)+1} - 1 = 2^k - 1 = n - 1\right)$

$= b.\,n\,\log n + (a + c)\,n - c$

$= O(n \log n)$ Worst case

Summary

- Sequential Approach
  - **Insertion Sort:**
    - Example
    - Time Complexity (Worst, Best, & Average Cases)
    - Space Complexity
    - Advantages and Disadvantages
- Divide-and-Conquer Approach
  - **Merge Sort:**
    - Example
    - Time Complexity (Worst Case)
    - Space Complexity
    - Advantages and Disadvantages