Theory of Algorithms – Homework 1 – Solution

Algo1

T (n) = c + T (3n/4)
= c + c + T ((3/4)^{2}n)
= c + c + c + T ((3/4)^{3}n)
…
T (n) = c + c + . . . + c + T ((3/4)^{k}n)

Stop when n = 1 ➔ (3/4)^{k}n = 1 ➔ (4/3)^k = n ➔ $k = \log_{4/3} n$

T (n) $= \sum_{i=1}^{k} c$
T (n) = c $\log_{4/3}$n ∈ $\Theta(\log_2 n)$

Algo2

$$Loops\ take: \sum_{i=1}^{\lfloor n/2 \rfloor} \sum_{j=1}^{\lfloor n/3 \rfloor} c_1 n = cn^3$$

T (n) = cn^3 + T (n – 5)
= cn^3 + c (n - 5)^3 + T (n – 10)
= cn^3 + c (n - 5)^3 + c (n - 10)^3 + T (n – 15)
…
T (n) = c (n – 0*5)^3 + c (n – 1*5)^3 + c (n – 2*5)^3 + c (n – 3*5)^3 + ... + c (n – (k-1)*5)^3 + T (n – k* 5)
Stop when n = 1 ➔ n – k * 5 = 1 ➔ k = n/5

T(n) = c $\sum_{i=1}^{k-1} (n - i * 5)^3 \leq (k-1) * n^3 = (\frac{n}{5} - 1) * n^3 = O(n^4)$

Algo3

T (n) = cn + 4T (n/4)
= cn + 4(cn/4 + 4T (n/4^2))
= cn + cn + 4^{2}T (n/4^2)
= cn + cn + 4^2(cn/4^2 + 4T (n/4^3))
= cn + cn + cn + 4^{3}T (n/4^3)
…
T (n) = cn + cn + cn + … + 4^{k}T (n/4^k)
Stop when n/4^k = 1 ➔ $\log_4 n$ = k

T (n) $= c \sum_{1}^{k} n = nk = n \log_4 n \in O\ (n \log_2 n)$

Algo4

$$Loops\ take: \sum_{i=1}^{n} c_1(\frac{n}{3}) = cn^2$$

T (n) $= cn^2 + T(2n/3)$
$= cn^2 + c(2n/3)^2 + T((2/3)^2 n)$
$= cn^2 + c(2n/3)^2 + c((2/3)^2 n)^2 + T((2/3)^3 n)$
…
$= cn^2 + c(2n/3)^2 + c((2/3)^2 n)^2 + \ldots + c((2/3)^{k-1} n)^{k-1} + T((2/3)^k n)$

Stop when $(2/3)^k n = 1$ ➔ $k = \log_{3/2} n$

T (n) $= cn^2 + c(2n/3)^2 + c((2/3)^2 n)^2 + \ldots + c((2/3)^{k-1} n)^{k-1} + T(1)$
$= cn^2 (1 + (2/3)^2 + (2/3)^4 + (2/3)^6 + \ldots + (2/3)^{k-1}) + c$
$\approx cn^2 (1 + (2/3)^2 + (2/3)^4 + (2/3)^6 + \ldots)$
$= cn^2(1/(1 - (2/3)^2)) = cn^2(9/5) \in \Theta(n^2)$

Algo5

$$Loops\ take: \sum_{i=1}^{\lfloor n/2 \rfloor} c_1(\frac{n}{2}) = cn^2$$

T (n) $= cn^2 + T(n - 5) + T(n - 8)$.
T (n) $= cn^2 + T(n - 5) + T(n - 8) \geq T(n - 5) + T(n - 8) \geq T(n - 8) + T(n - 8)$
$= 2T(n - 8) \geq 2^2 T(n - 16) \geq 2^3 T(n - 24) \geq 2^{n/8} T(1) = c2^{n/8} \in \Omega(2^{n/8})$

Algo6

T (n) $= c\sqrt{n} + T(n - 3)$
$= c\sqrt{n} + c\sqrt{n-3} + T(n - 6)$
$= c\sqrt{n} + c\sqrt{n-3} + c\sqrt{n-6} + T(n - 9)$
…
T (n) $= c\sqrt{n - 0 * 3} + c\sqrt{n - 1 * 3} + c\sqrt{n - 2 * 3} + \ldots + c\sqrt{n - (k-1) * 3} + T(n - k*3)$
Stop when $1 = n - k*3$ ➔ $k = n/3$

T (n) $= c\sum_{i=0}^{k-1} \sqrt{n - i * 3} \leq c\sum_{i=0}^{k-1} \sqrt{n} = c\sqrt{n}\,(n/3) = O(n^{1.5})$

Algo7

T (n) $= cn + T(n-3) + T(n-7) + T(n-11) + \ldots + T(1)$
$\geq T(n-3) + T(n-7) \geq T(n-7) + T(n-7)$
$= 2T(n-7) \geq 2^2 T(n-14) \geq 2^3 T(n-21) \geq 2^{n/7} T(1) \in \Omega(2^{n/7})$

Algo8

$$\textit{Loops take}: \sum_{i=1}^{4} \sum_{j=1}^{n-1} c_1 n/2 = cn^2$$

T (n) $= cn^2 + 4T(n/2)$
$= cn^2 + 4(c(n/2)^2 + 4T(n/2^2))$
$= cn^2 + cn^2 + 4^2 T(n/2^2)$
$= cn^2 + cn^2 + 4^2(c(n/2^2)2 + 4T(n/2^3))$
$= cn^2 + cn^2 + cn^2 + 4^3 T(n/2^3)$
…
T (n) $= cn^2 + cn^2 + cn^2 + \ldots + cn^2 + 4^k T(n/2^k)$
Stop when n= 1 ➔ $n/2^k = 1$ ➔ $k = \log_2 n$

T (n) $= \sum_{i=0}^{\log_2 n-1} cn^2 + 4^{\log_2 n} T(1) = \sum_{i=0}^{\log_2 n} cn^2 = \Theta(n^2 \log_2 n)$

Algo9

T (n) $= T(n-2) + T(n-6) + T(n-18) + T(n-54) + \ldots + T(n - 2 \times 3^k) + \ldots$
$\geq T(n-2) + T(n-6) \geq T(n-6) + T(n-6) = 2T(n-6)$.
T (n) $\geq 2T(n-6) \geq 2 \times 2 \times T(n-12) \geq 2 \times 2 \times 2 \times T(n-18)$
$\geq \sum_{n}^{n/6} 2 + T(1) = 2^{n/6} c = \Omega(2^{n/6})$