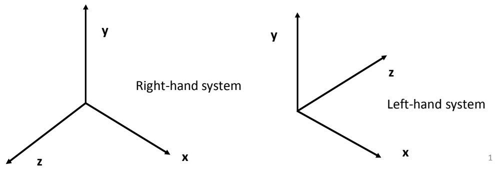
## Basic Entities

- Coordinate system: has an origin and some mutually perpendicular axes emanating from the origin.
- Point P: a location in space


- Vector v : with length and direction, physical entities, such as force, and velocity. Vector has no fixed location, seen as points displacement

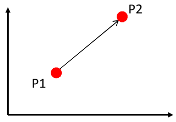




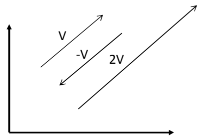

## Vector-scalar multiplication

- $V = P_2 - P_1$

$$= \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$



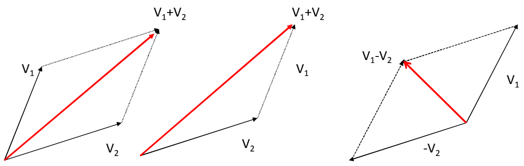

- Vector permits two fundamental operations: add them, multiply them with real number
- $aV = \begin{bmatrix} aV_x \\ aV_y \\ aV_z \end{bmatrix}$






## Vector addition

- Sum of two vectors
- Subtraction of two vectors
- Adding and subtraction of corresponding components of two vectors gives a new vector






## Normalize a vector

- Magnitude (length): the distance from the tail to the head.

$$|V| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

- Normalization: Scale a vector to have a unity length, unit vector

$$\hat{V} = \frac{V}{|V|}$$



## Example

- Given a vector $V = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ then $2V = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix}$
- Given vector $r = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $s = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$
- $r + s = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$
- $r - s = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + -1 * \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$
- $|r| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$
- $\hat{r} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

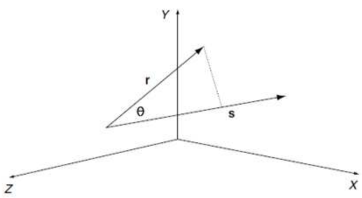


## Dot Product (Scalar Product)

- We obtain a scalar value from two vectors

$V_1 . V_2 = |V_1|\ |V_2| \cos\theta$

$V_1 . V_2 = V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1z}V_{2z}$



## Example

Given vector r = $\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$ and s = $\begin{bmatrix} 5 \\ 6 \\ 10 \end{bmatrix}$

|r| = =$\sqrt{2^2 + (-3)^2 + 4^2}$ = $\sqrt{29}$ = 5.385

|s| = =$\sqrt{5^2 + 6^2 + 10^2}$ = $\sqrt{161}$ = 12.689

|r| |s| cos θ = 2 * 5 + (-3) * 6 + 4 * 10 = 32

5.385 * 12.689 * cos θ = 32

cos θ = 32 / (5.385 * 12.689) = 0.468

Θ = $\cos^{-1}$(0.468 ) = 62.1

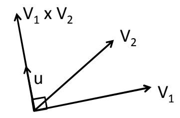


## Cross Product

- Combine to vectors to produce another vector

  $V_1 \times V_2 = u|V_1|\ |V_2| \sin\theta$

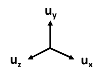
u is a unit vector that is perpendicular to both $V_1$ and $V_2$




$V_1 \times V_2 = - V_2 \times V_1$

$V_1 \times V_2 = \begin{bmatrix} V_{1y}V_{2z} - V_{1z}V_{2y} \\ V_{1z}V_{2x} - V_{1x}V_{2z} \\ V_{1x}V_{2y} - V_{1y}V_{2x} \end{bmatrix}$

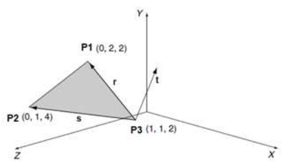
$V_1 \times V_2 = \begin{bmatrix} u_x & u_y & u_z \\ V_{1x} & V_{1y} & V_{1z} \\ V_{2x} & V_{2y} & V_{2z} \end{bmatrix}$






## Example: Unit Normal Vector for a Triangle

r = $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and s = $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$




r x s = $\begin{bmatrix} u_x & u_y & u_z \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

r x s = $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ = t

|t| = $\sqrt{2^2 + 2^2 + 1^2}$ = 3➔ t is not a unit vector

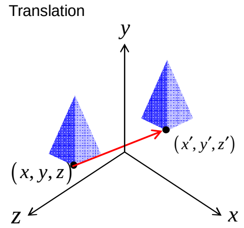
$\hat{t}$ = $\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$



## 3D Transformations

Very similar to 2D. Using 4x4 matrices rather than 3x3.

Translation




$x' = x + t_x$

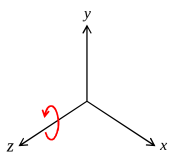
$y' = y + t_y$

$z' = z + t_z$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



## 3D Rotation



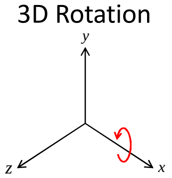

Z-axis rotation

x′ = x cosθ – y sinθ

y′ = x sinθ + y cosθ

z′ = z

$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$



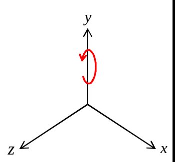

X-axis rotation

y′ = y cosθ – z sinθ

z′ = y sinθ + z cosθ

x′ = x

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$




Y-axis rotation

x′ = z sinθ + x cosθ

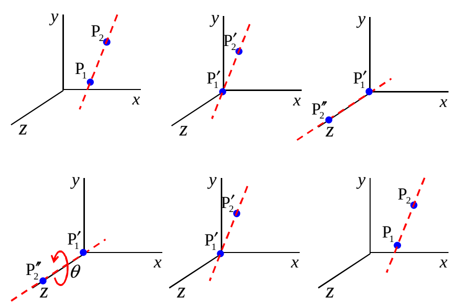
z′ = z cosθ - x sinθ

y′ = y

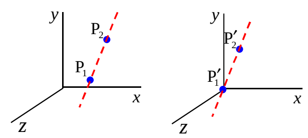
$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

## General 3D Rotation

1. Translate the object such that rotation axis passes through the origin.
2. Rotate the object such that rotation axis coincides with one of Cartesian axes.
3. Perform specified rotation about the Cartesian axis.
4. Apply inverse rotation to return rotation axis to original direction.
5. Apply inverse translation to return rotation axis to original position.

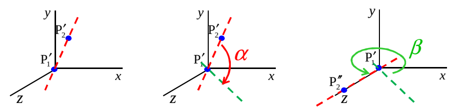





$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotate around x axis, then rotate around y axis




$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha & 0 \\ 0 & sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} cos\beta & 0 & sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\beta & 0 & cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



# Overall Transformations

$$\begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha & 0 \\ 0 & sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} cos\beta & 0 & sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\beta & 0 & cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\cdot \begin{bmatrix} cos\beta & 0 & sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -sin\beta & 0 & cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha & 0 \\ 0 & sin\alpha & cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The only unknowns are the angles α and β.



The vector from $\mathbf{P}_1$ to $\mathbf{P}_2$ is:

$\mathbf{V} = \mathbf{P}_2 - \mathbf{P}_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

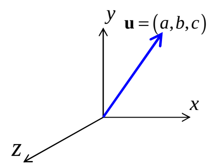
Unit rotation vector: $\mathbf{u} = \mathbf{V} / |\mathbf{V}| = (a, b, c)$

$a = (x_2 - x_1) / |\mathbf{V}|$

$b = (y_2 - y_1) / |\mathbf{V}|$

$c = (z_2 - z_1) / |\mathbf{V}|$

$\sqrt{a^2 + b^2 + c^2} = 1$






Rotating $\mathbf{u}$ to coincide with $z$ axis

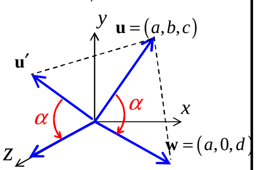
First rotate $\mathbf{u}$ around $x$ axis to lay in $x - z$ plane.

Equivqlent to rotation $\mathbf{u}$'s projection on $y - z$ plane around $x$ axis.

$\cos\alpha = c / \sqrt{b^2 + c^2} = c/d$, $\sin\alpha = b/d$.

We obtained a unit vector $\mathbf{w} = \left(a, 0, \sqrt{b^2 + c^2} = d\right)$ in $x - z$ plane.

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$






u = (a, b, c)
u' = (0, b, c)
$u_z$ = (0, 0, 1) unit vector in Z direction
$u_x$ = (1, 0, 0) unit vector in X direction

$$\cos\alpha = \frac{u'.u_z}{|u'||u_z|}$$

$$\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

$$\sin\alpha = \frac{u' \times u_z}{u_x|u'||u_z|} = \frac{u_x b}{u_x d} = \frac{b}{d}$$

$$u' \times u_z = \begin{bmatrix} u_x & u_y & u_z \\ 0 & b & c \\ 0 & 0 & 1 \end{bmatrix}$$

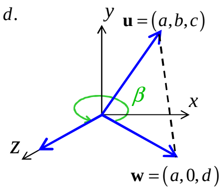
Rotate $\mathbf{w}$ counterclockwise around $y$ axis.

$\mathbf{w}$ is a unit vector whose $x-$component is $a$, $y-$component is 0,

hence $z-$component is $\sqrt{b^2+c^2}=d$.

$\cos\beta = d,\quad \sin\beta = -a$

$$\mathbf{R}_y(\beta)=\begin{bmatrix} d & 0 & a & 0\\ 0 & 1 & 0 & 0\\ -a & 0 & d & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$




$$\mathbf{R}(\theta)=\mathbf{T}^{-1}\cdot\mathbf{R}_x^{-1}(\alpha)\cdot\mathbf{R}_y^{-1}(\beta)\cdot\mathbf{R}_z(\theta)\cdot\mathbf{R}_y(\beta)\cdot\mathbf{R}_x(\alpha)\cdot\mathbf{T}$$

19

w = (a, 0, d)

$u_z$ = (0, 0, 1) unit vector in Z direction

$$\cos\beta = \frac{w.u_z}{|w||u_z|} = \text{d}$$

$$\sin\beta = \frac{w\times u_z}{u_y|w||u_z|} = \frac{u_x(-a)}{u_x} = -a$$

$$w\times u_z = \begin{bmatrix} u_x & u_y & u_z\\ a & 0 & d\\ 0 & 0 & 1\end{bmatrix}$$

20

$$\mathbf{R}_z(\theta)=\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

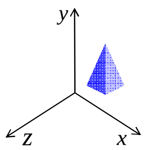
$$\mathbf{R}(\theta)=\mathbf{T}^{-1}\cdot\mathbf{R}_x^{-1}(\alpha)\cdot\mathbf{R}_y^{-1}(\beta)\cdot\mathbf{R}_z(\theta)\cdot\mathbf{R}_y(\beta)\cdot\mathbf{R}_x(\alpha)\cdot\mathbf{T}$$

$$\mathbf{M}_R(\theta)=$$

$$\begin{bmatrix} a^2(1-\cos\theta)+\cos\theta & ab(1-\cos\theta)-c\sin\theta & ac(1-\cos\theta)+b\sin\theta\\ ba(1-\cos\theta)+c\sin\theta & b^2(1-\cos\theta)+\cos\theta & bc(1-\cos\theta)-a\sin\theta\\ ca(1-\cos\theta)-b\sin\theta & cb(1-\cos\theta)+a\sin\theta & c^2(1-\cos\theta)+\cos\theta\end{bmatrix}$$

21

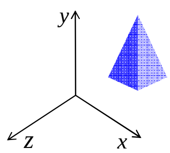
## 3D Scaling




$x' = x\cdot S_x$

$y' = y\cdot S_y$

$z' = x\cdot S_z$

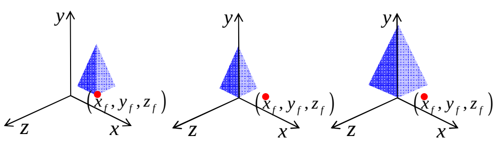


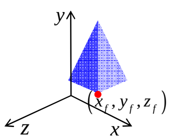

Enlarging object also moves it from origin

$$\mathbf{P}'=\begin{bmatrix} x'\\ y'\\ z'\\ 1\end{bmatrix}=\begin{bmatrix} S_x & 0 & 0 & 0\\ 0 & S_y & 0 & 0\\ 0 & 0 & S_z & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix} x\\ y\\ z\\ 1\end{bmatrix}=\mathbf{S}\cdot\mathbf{P}$$

22

Scaling with respect to a fixed point (not necessarily of object)




$$\mathbf{T}\cdot\mathbf{S}\cdot\mathbf{T}^{-1}=\begin{bmatrix} S_x & 0 & 0 & (1-S_x)x_f\\ 0 & S_y & 0 & (1-S_y)y_f\\ 0 & 0 & S_z & (1-S_z)z_f\\ 0 & 0 & 0 & 1\end{bmatrix}$$

23