Chapter 3: The Solution of Linear Systems $AX = B$

3.1 Introduction to Vectors and Matrices

A real N-dimensional vector X is an ordered set of N real numbers:

$$X = (x_1, x_2, \cdots, x_N)$$

Here the numbers $x_1, x_2, \cdots, x_N$ are called the **components of X**.

Let another vector be $Y = (y_1, y_2, \cdots, y_N)$. The two vectors X and Y are equal if:

$$X = Y \qquad \text{if and only if} \qquad x_j = y_j \qquad \text{for } j = 1, 2, \cdots, N$$

The sum of vectors X and Y:

$$X + Y = (x_1 + y_1, x_2 + y_2, \cdots, x_N + y_N)$$

The negative of the vector X:

$$-X = (-x_1, -x_2, \cdots, -x_N)$$

The difference $Y - X$:

$$Y - X = Y + (-X) = (y_1 - x_1, y_2 - x_2, \cdots, y_N - x_N)$$

If c is a real number (scalar), we define **scalar multiplication** cX as:

$$cX = (cx_1, cx_2, \cdots, cx_N)$$

The **dot product** of the two vectors X and Y is a scalar quantity:

$$X.Y = x_1y_1 + x_2y_2 + \cdots + x_Ny_N$$

The norm (or length) of the vector X:

$$\|X\| = \sqrt{(x_1^2 + x_2^2 + \cdots + x_N^2)}$$

if X and Y are position vectors that locate points, then

$Y - X$ (displacement form position X to position Y)

The distance between two points in N-space:

$$\|Y - X\| = \sqrt{((y_1^2 - x_1^2) + (y_2^2 - x_2^2) + \cdots + (y_N^2 - x_N^2))}$$

We say that the points lie in N-**dimensional Euclidean Space**.

The set of vectors has a zero element **0**, which is defined by

$$0 = (0, 0, \cdots, 0)$$

It's useful to write vectors as columns

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

The linear combination $cX + dY$

$$cX + dY = \begin{bmatrix} cx_1 + dy_1 \\ cx_2 + dy_2 \\ \vdots \\ cx_N + dy_N \end{bmatrix}$$

We use the superscript " ' " for transpose to indicate that a row vector should be converted to a column vector and vice versa.

$$(x_1, x_2, \cdots, x_N)' = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}' = (x_1, x_2, \cdots, x_N)$$

Example: Let $X = (2, -3, 5, -1)$ and $Y = (6, 1, 2, -4)$ vectors in 4-space.

Sum	$X + Y = (8, -2, 7, -5)$
Difference	$X - Y = (-4, -4, 3, 3)$
Scalar multiple	$3X = (6, -9, 15, -3)$
Length	$\|X\| = \sqrt{(4 + 9 + 25 + 1)} = \sqrt{39}$
Dot product	$X.Y = 12 - 3 + 10 + 4 = 23$
Displacement form X to Y	$X - Y = (4, 4, -3, -3)$
Distance form X to Y	$\|X - Y\| = \sqrt{(16 + 16 + 9 + 9)} = \sqrt{50}$

Theorem (Vector Algebra) Suppose that X, Y, and Z are N-dimensional vectors and a and b are scalars

$X + Y = Y + X$	commutative property
$0 + X = X + 0$	additive identity
$X - X = X + (-X) = 0$	additive inverse
$(X + Y) + Z = X + (Y + Z)$	associative property
$(a + b)X = aX + bX$	distributive property for scalars
$a(X + Y) = aX + aY$	distributive property for vectors
$a(bX) = (ab)X$	associative property for scalars

Matrices and Two-dimensional Arrays

A matrix having M rows and N columns is called MxN matrix.

$$A = [a_{ij}]_{MxN} \qquad \text{for } 1 \leq i \leq M,\ 1 \leq j \leq N$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1N} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{iN} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{M1} & \cdots & a_{Mj} & \cdots & a_{MN} \end{bmatrix}$$

Theorem (Matrix Addition) Suppose that A, B, and C are MxN matrices and p and q are scalars

$B + A = A + B$	commutative property
$0 + A = A + 0$	additive identity
$A - A = A + (-A) = 0$	additive inverse
$(A + B) + C = A + (B + C)$	associative property
$(p + q)A = pA + qA$	distributive property for scalars
$p(A + B) = pA + pB$	distributive property for matrices
$p(qA) = (pq)A$	associative property for scalars

3.2 Properties of Vectors and Matrices

Matrix Multiplication

if $A = [a_{ik}]_{MxN}$ and $B = [b_{kj}]_{NxP}$ are two matrices with the property that A has as many columns as B has rows, then AB is defined to be the matrix C of dimensions MxP

$$AB = C = [c_{ij}]_{MxP}$$

where the element c_{ij} is the dot product of the ith row of A and the jth column of B

$$c_{ij} = \sum_{k=1}^{N} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{iN}b_{Nj}$$

for $i = 1, 2, ..., M$ and $j = 1, 2, ..., P$

Example:

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 8 & -6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 3 & 8 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 10+9 & -4+24 & 2-18 \\ -5+12 & 2+32 & -1-24 \end{bmatrix} = \begin{bmatrix} 19 & 20 & -16 \\ 7 & 34 & -25 \end{bmatrix} = C$$

A linear combination of the variable $x_1, x_2, \cdots, x_N$ is the sum

$$a_1x_1 + a_2x_2 + \cdots + a_Nx_N$$

A linear combination is required to take on a prescribed value b

$$a_1x_1 + a_2x_2 + \cdots + a_Nx_N = b$$

System of linear equations with M equations and N unknowns are given:

$$\begin{gathered}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = b_1 \\
\vdots \\
a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kN}x_N = b_k \\
\vdots \\
a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N = b_M
\end{gathered}$$

A linear system of equations can be represented as matrix product:

$$AX = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1N} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kj} & \cdots & a_{kN} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{M1} & \cdots & a_{Mj} & \cdots & a_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ \vdots \\ b_M \end{bmatrix} = B$$

Some Special Matrices

Zero matrix of dimensions MxN is denoted by

$$0 = [0]_{MxN}$$

Identity matrix of order N is the square matrix given by

$$I_N = [\delta_{ij}]_{NxN} \text{ where } \delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases}$$

An NxN matrix A is called **nonsingular** or invertible if there exists an NxN matrix B such that

$$AB = BA = I$$

If no such matrix B can be found, A is said to **singular**. When B can be found, we usually write $B = A^{-1}$.

$$AA^{-1} = A^{-1}A = I \text{ if } A \text{ is nonsingular}$$

The determinant of a square matrix A is a scalar quantity del(A) or $|A|$

$$det(A) = \begin{vmatrix} a_{11} & \cdots & a_{12} & \cdots & a_{1N} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ a_{N1} & \cdots & a_{N2} & \cdots & a_{NN} \end{vmatrix}$$

if $A = [a_{ij}]$ is $1x1$ matrix, we define $del(A) = a_{11}$

if $A = [a_{ij}]_{NxN}$ where $N \geq 2$, then let M_{ij} be the determinant of the $N-1xN-1$ submatrix of A obtained by deleting the ith row and jth column of A. The **cofactor** A_{ij} is defined as $A_{ij} = (-1)^{i+j}M_{ij}$.

$$det(A) = \sum_{j=1}^{N} a_{ij}A_{ij} \text{ (ith row expansion)}$$

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Example:

$$A = \begin{vmatrix} 2 & 3 & 8 \\ -4 & 5 & -1 \\ 7 & -6 & 9 \end{vmatrix}$$

$$det(A) = (2)\begin{vmatrix} 5 & -1 \\ -6 & 9 \end{vmatrix} - (3)\begin{vmatrix} -4 & -1 \\ 7 & 9 \end{vmatrix} + (8)\begin{vmatrix} -4 & 5 \\ 7 & -6 \end{vmatrix}$$

$$det(A) = (2)(45 - 6) - (3)(-36 + 7) + (8)(24 - 35) = 77.$$

Theorem. Assume that A is an NxN matrix. The following statements are equivalent.

- Given any Nx1 matrix B, the linear system $AX = B$ has a unique solution.
- The matrix A is nonsingular (i.e. $A^{-1}exists$).
- The system of equations $AX = 0$ has the unique solution $X = 0$.
- $det(A) \neq 0$.
- $AX = B$ implies $A^{-1}AX = A^{-1}B$, which implies $X = A^{-1}B$

3.3 Upper-triangle Linear Systems

Definition. an NxN matrix $A = [a_{ij}]$ is called **upper triangular** provided that elements satisfy $a_{ij} = 0$ whenever $i > j$. The NxN matrix $A = [a_{ij}]$ is called **lower triangular** provided that $a_{ij} = 0$ whenever $i < j$

$AX = B$ is said to be upper triangular system of linear equations if it has the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1N-1}x_{N-1} + a_{1N}x_N &= b_1 \\
a_{22}x_2 + a_{23}x_3 + \cdots + a_{2N-1}x_{N-1} + a_{2N}x_N &= b_2 \\
a_{33}x_3 + \cdots + a_{3N-1}x_{N-1} + a_{3N}x_N &= b_3 \\
\vdots \qquad & \qquad \vdots \\
a_{N-1N-1}x_{N-1} + a_{N-1N}x_N &= b_{N-1} \\
a_{NN}x_N &= b_N
\end{aligned}$$

Theorem (Back Substitution) Suppose that $AX = B$ is an upper-triangular system. If

$$a_{kk} \neq 0 \text{ for } k = 1, 2, ..., N,$$

then there exists a unique solution to the system.

Proof: The last equation involves only x_N so

$$x_N = \frac{b_N}{a_{NN}}$$

Now x_N is known and it can be used in the next-to-last equation:

$$x_{N-1} = \frac{b_{N-1} - a_{N-1N}x_N}{a_{N-1}a_{N-1}}$$

Now x_N and x_{N-1} are used to find x_{N-2}

$$x_{N-2} = \frac{b_{N-2} - a_{N-2N-1}x_{N-1} - a_{N-2N}x_N}{a_{N-2}a_{N-2}}$$

Once the values $x_N, x_{N-1}, ..., x_{k+1}$ are known, the general step is:

$$x_k = \frac{b_k - \sum_{j=k+1}^{N} a_{kj}x_j}{a_k a_k} \text{ for } k = N-1, N-2, ..., 1.$$

Example: Use back substitution to solve the linear system

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\ -2x_2 + 7x_3 + 4x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6 \end{aligned}$$

$$x_4 = \frac{6}{3} = 2$$

$$x_3 = \frac{4-5(2)}{6} = -1$$

$$x_2 = \frac{-7-7(-1)+4(2)}{-2} = -4$$

$$x_1 = \frac{20+1(-4)-2(-1)-3(2)}{4} = 3$$

Example: Show that there is no solution to the linear system

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\ 0x_2 + 7x_3 - 4x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6 \end{aligned}$$

We have $x_4 = \frac{6}{3} = 2$, which is substituted into the second and third equations to obtain

$$7x_3 - 8 = -7$$

$$6x_3 + 10 = 4$$

The first equation implies $x_3 = 1/7$, and the second equation implies $x_3 = -1$. This contradiction leads to the conclusion that there is no solution.

Example: Show that there is infinitely many solutions to

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\ 0x_2 + 7x_3 + 0x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6 \end{aligned}$$

$x_4 = 2$ and $x_3 = -1$. Only x_4 and x_3 have obtained from the second through forth equations.

$$x_2 = 4x_1 - 16$$

which has infinitely many solutions. If we choose a value for x_1 then the value for x_2 is uniquely determined. For example, if $x_1 = 2$ then $x_2 = -8$.

The linear system $AX = B$, where A is an NxN matrix, has a unique solution if and only if $det(A) \neq 0$

Theorem. if the NxN matrix $A = [a_{ij}]$ is either upper or lower triangular, then

$$det(A) = a_{11}a_{22}a_{33}\cdots a_{NN} = \prod_{i=1}^{N} a_{ii}$$

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function X=BackSubstitution (A,B)

%Input  - A is an n x n upper-triangular nonsingular matrix
%       - B is an n x 1 matrix
%Output - X is the solution to the linear system AX = B

 n = length(B);
 X = zeros(n,1);
 X(n) = B(n) / A(n,n);

 for k = n-1:-1:1
    X(k) = (B(k)-A(k,k+1:n)*X(k+1:n))/A(k,k);
 end
```