3.4 Gaussian Elimination and Pivoting

To solve a general system $AX = B$ with n equations and n unknowns, construct an equivalent upper-triangular system $UX = Y$ that can be solved using Back-substitution.

Theorem (Elementary Transformation). The following operations applied to a linear system yield an equivalent system:

1. Interchanges: The order of 2 equations can be changed.
2. Scaling: Multiplying an equation by nonzero constant.
3. Replacement: An equation can be replaced by the sum of itself and a nonzero multiple of any other equation.

Example: Find the parabola $y = A + Bx + Cx^2$ that passes through the points $(1, 1)$, $(2, -1)$ and $(3, 1)$.
For each point we obtain an equation

$$\begin{array}{rcrcrcrcl} A & + & B & + & C & = & 1 & at(1,1) & Q_1 \\ A & + & 2B & + & 4C & = & -1 & at(2,-1) & Q_2 \\ A & + & 3B & + & 9C & = & 1 & at(3,1) & Q_3 \end{array}$$

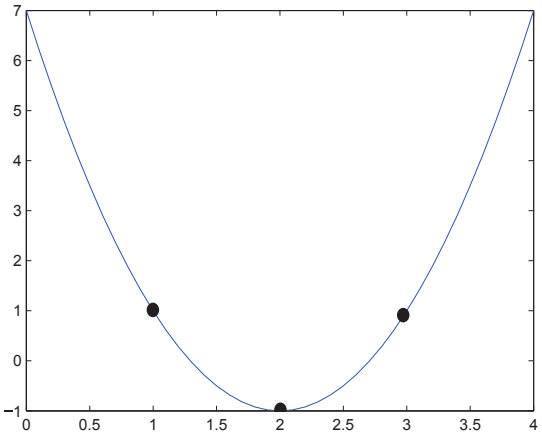
A is eliminated from the second and third equation by subtracting the first equation from them

$$\begin{array}{rcrcrcrcl} A & + & B & + & C & = & 1 & & Q_1 \\ & & B & + & 3C & = & -2 & & Q_2 - Q_1 \\ & & 2B & + & 8C & = & 0 & & Q_3 - Q_1 \end{array}$$

B is eliminated from the third equation by subtracting from it two times the second equation

$$\begin{array}{rcrcrcrcl} A & + & B & + & C & = & 1 & & Q_1 \\ & & B & + & 3C & = & -2 & & Q_2 \\ & & & & 2C & = & 4 & & Q_3 - 2Q_2 \end{array}$$

Using back-substitution, $C = 2$, $B = -8$, and $A = 7$



Augmented matrix is $NxN+1$ matrix in which the coefficients of B are stored in column $N+1$. The augmented matrix is donated $[A|B]$ and the linear system is represented as:

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & & \cdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} & b_N \end{array}\right]$$

Definition (Pivot). The number a_{rr} in the coefficient matrix A that is used to eliminate a_{kr}, where $k = r+1, r+2, ..., N$, is called the rth **pivotal element**, and the rth row is called the **pivot row**.

Example: Express the following matrix in augmented matrix form and find an equivalent upper-triangular system and the solution.

$$\begin{array}{rcrcrcrcll} x_1 & + & 2x_2 & + & x_3 & + & 4x_4 & = & 13 & Q_1 \\ 2x_1 & + & 0x_2 & + & 4x_3 & + & 3x_4 & = & 28 & Q_2 \\ 4x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & = & 20 & Q_3 \\ -3x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & = & 6 & Q_4 \end{array}$$

The augmented matrix is

$$\begin{array}{rc} pivot \rightarrow & \left[\begin{array}{cccc|c} \underline{1} & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right] \\ m_{21} = 2 & \\ m_{31} = 4 & \\ m_{41} = -3 & \end{array} \qquad \begin{array}{c} Q_1 \\ Q_2 - 2Q_1 \\ Q_3 - 4Q_1 \\ Q_4 - -3Q_1 \end{array}$$

The first row is used to eliminate elements in the first column below the diagonal. The values m_{k1} are the multiplies of row 1 that are to be subtracted from row k for $k = 2, 3, 4$.

$$\begin{array}{c} \\ pivot \rightarrow \\ m_{32} = 1.5 \\ m_{42} = -1.75 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & \underline{-4} & 2 & -5 & 2 \\ 0 & -6 & -2 & -15 & -32 \\ 0 & 7 & 6 & 14 & 45 \end{array}\right] \begin{array}{c} Q_1 \\ Q_2 \\ Q_3 - 1.5Q_2 \\ Q_4 - -1.75Q_2 \end{array}$$

The second row is used to eliminate elements in the second column below the diagonal.

$$\begin{array}{c} \\ \\ pivot \rightarrow \\ m_{43} = -1.9 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & \underline{-5} & -7.5 & -35 \\ 0 & 0 & 9.5 & 5.25 & 48.5 \end{array}\right] \begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 - -1.9Q_3 \end{array}$$

Finally, the multiplies $m_{43} = -1.9$ of the third row is subtracted from the fourth row.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array}\right]$$

back-substitution is used to get

$$x_4 = 2,\ x_3 = 4,\ x_2 = -1, \text{ and } x_1 = 3 \quad .$$

This process is called **Gaussian elimination**.

If $a_{kk} = 0$, row k cannot be used to eliminate elements in column k, and row k must be interchanged with some row below the diagonal to obtain nonzero pivot.

If this cannot be done, the system does not have a unique solution.

Theorem (Gaussian Elimination with Back Substitution). If A is an NxN nonsingular matrix, then there exists a system $UX = Y$, equivalent to $AX = B$, where U is upper-triangular matrix with $a_{uu} \neq 0$. After U and Y are constructed, back substitution can be used to solve $UX = Y$ for X

Pivoting to avoid $a_{pp} = 0$

If $a_{pp} = 0$, row p cannot be used to eliminate the elements in column p below the main diagonal.
It is necessary to find row k, where $a_{kp} \neq 0$ and $k > p$, and then interchange row p and row k. This process is called **pivoting**.

- Trivial pivoting: If $a_{pp} = 0$, locate the first row below p in which $a_{kp} \neq 0$ and switch rows k and p.
- Partial pivoting: Locate row k in which the element has the largest absolute value.
  $$|a_{kp}| = max\{|a_{pp}|, |a_{p+1p}|, ..., |a_{N-1p}|, |a_{Np}|\}$$
  and switch row p with row k if $k > p$.

```
function X = UpperTriBackSub(A,B)
%Input  - A is an N x N nonsingular matrix
%      - B is an N x 1 matrix
%Output - X is an N x 1 matrix containing the solution to AX=B.

%Initialize X and the temporary storage matrix C
[N N] = size(A);
X = zeros(N,1);
C = zeros(1,N+1);

%Form the augmented matrix: Aug=[A|B]
Aug = [A B];

for p = 1:N-1
    %Partial pivoting for column p
    [Y,j] = max(abs(Aug(p:N,p)));
    %Interchange row p and j
    C = Aug(p,:);
    Aug(p,:) = Aug(j+p-1,:);
    Aug(j+p-1,:) = C;

    if Aug(p,p)== 0
        'A was singular.  No unique solution'
        break
    end

    %Elimination process for column p
    for k=p+1:N
        m = Aug(k,p)/Aug(p,p);
        Aug(k,p:N+1) = Aug(k,p:N+1)- m * Aug(p,p:N+1);
    end
end

%Back Substitution on [U|Y] using Program 3.1
X = BackSubstitution (Aug(1:N,1:N),Aug(1:N,N+1));
```

Practice

Find and plot the cubic $y = A + Bx + Cx^2 + Dx^3$ that passes through (0, 0), (1, 1), (2, 2), and (3, 2)

Solve the linear systems:

$$\begin{aligned} 2x_1 &+ 4x_2 - 6x_3 = -4 \\ x_1 &+ 5x_2 + 3x_3 = 10 \\ x_1 &+ 3x_2 + 2x_3 = 15 \end{aligned}$$

$$\begin{aligned} 2x_1 &- 3x_2 + 100x_3 = 1 \\ x_1 &+ 10x_2 - 0.001x_3 = 0 \\ 3x_1 &- 100x_2 + 0.01x_3 = 0 \end{aligned}$$

3.6 Iterative Methods for Linear Systems

We are going to extend some of the iterative methods in chapter 2 to higher dimensions. Consider an extensions of fixed-point iteration applied to systems of linear equations

Jacobi Iteration

Consider the system

$$\begin{array}{rcrcrcr} 4x & - & y & + & z & = & 7 \\ 4x & - & 8y & + & z & = & -21 \\ -2x & + & y & + & 5z & = & 15 \end{array}$$

These equations can be written in the form

$$x = \frac{7+y-z}{4}$$

$$y = \frac{21+4x+z}{8}$$

$$z = \frac{15+2x-y}{5}$$

The Jacobi iterations are

$$x_{k+1} = \frac{7+y_k-z_k}{4}$$

$$y_{k+1} = \frac{21+4x_k+z_k}{8}$$

$$z_{k+1} = \frac{15+2x_k-y_k}{5}$$

If we start with $P_0 = (x_0, y_0, z_0) = (1, 2, 2)$, then the iteration appears to converge to the solution $(2, 4, 3)$

Substitute $x_0 = 1, y_0 = 2, z_0 = 2$

$$x_1 = \frac{7+2-2}{4} = 1.75$$

$$y_1 = \frac{21+4+2}{8} = 3.375$$

$$z_1 = \frac{15+2-2}{5} = 3.00$$

Table 3.2 Convergent Jacobi Iteration for the Linear System (1)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.375	3.0
2	1.84375	3.875	3.025
3	1.9625	3.925	2.9625
4	1.99062500	3.97656250	3.00000000
5	1.99414063	3.99531250	3.00093750
⋮	⋮	⋮	⋮
15	1.99999993	3.99999985	2.99999993
⋮	⋮	⋮	⋮
19	2.00000000	4.00000000	3.00000000

Sometime Jacobi iteration does not work.

Let the linear system be rearranged as follows:

$$\begin{array}{rcrcrcr} -2x & + & y & + & 5z & = & 15 \\ 4x & - & 8y & + & z & = & -21 \\ 4x & - & y & + & z & = & 7 \end{array}$$

The equations can be written in the form

$$x = \frac{-15+y+5z}{2}$$

$$y = \frac{21+4x+z}{8}$$

$$z = 7 - 4x + y$$

The Jacobi iterations are

$$x_{k+1} = \frac{-15+y_k+5z_k}{2}$$

$$y_{k+1} = \frac{21+4x_k+z_k}{8}$$

$$z_{k+1} = 7 - 4x_k + y_k$$

If we start with $P_0 = (x_0, y_0, z_0) = (1, 2, 2)$, then the iteration will diverge away from the solution $(2, 4, 3)$

Substitute $x_0 = 1, y_0 = 2, z_0 = 2$

$$x_1 = \frac{-15+2+10}{2} = -1.5$$
$$y_1 = \frac{21+4+2}{8} = 3.375$$
$$z_1 = 7 - 4 + 2 = 5.00$$

Table 3.3 Divergent Jacobi Iteration for the Linear System (4)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	−1.5	3.375	5.0
2	6.6875	2.5	16.375
3	34.6875	8.015625	−17.25
4	−46.617188	17.8125	−123.73438
5	−307.929688	−36.150391	211.28125
6	502.62793	−124.929688	1202.56836
⋮	⋮	⋮	⋮

Gauss-Seidel Iteration

To speed up convergence, x_{k+1} can be used in place of x_x in the computation of y_{k+1}, Similarly, x_{k+1} and y_{k+1} might be used in the computation of z_{k+1}

$$x_{k+1} = \frac{7+y_k-z_k}{4}$$
$$y_{k+1} = \frac{21+4x_{k+1}+z_k}{8}$$
$$z_{k+1} = \frac{15+2x_{k+1}-y_{k+1}}{5}$$

If we start with $P_0 = (x_0, y_0, z_0) = (1, 2, 2)$, then the iteration appears to converge to the solution $(2, 4, 3)$

Substitute $x_0 = 1, y_0 = 2, z_0 = 2$

$$x_1 = \frac{7+2-2}{4} = 1.75$$
$$y_1 = \frac{21+4(1.75)+2}{8} = 3.75$$
$$z_1 = \frac{15+2(1.75)-3.75}{5} = 2.95$$

Definition. A matrix A of dimension NxN is said to be strictly diagonally dominant provided that

$$|a_{kk}| > \sum_{j=1, j\neq k}^{N} |a_{ij}| \qquad \text{for } k = 1, 2, ..., N$$

This means that in each row in the matrix the magnitude of the element on the main diagonal must exceed the sum of the magnitude of all other elements in the row.

In the Example

row 1: $|4| > |-1| + |1|$
row 2: $|-8| > |4| + |1|$
row 3: $|5| > |-2| + |1|$

Suppose that a linear system is

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1N}x_N &= b_1 \\
&\vdots \\
a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jj}x_j + \cdots + a_{jN}x_N &= b_j \\
&\vdots \\
a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{Nj}x_j + \cdots + a_{NN}x_N &= b_N
\end{aligned}$$

Let the kth point be $P_k = (x_1^k, ..., x_j^k, ..., x_N^k)$; then the next point is $P_{k+1} = (x_1^{k+1}, ..., x_j^{k+1}, ..., x_N^{k+1})$

Jacobi iteration:

$$x_j^{k+1} = \frac{b_j - a_{j1}x_1^k - ... - a_{jj-1}x_{j-1}^k - a_{jj+1}x_{j+1}^k - ... - a_{jN}x_N^k}{a_{jj}}$$

for $k = 1, 2, ..., N$

Gauss-Seidel iteration:

$$x_j^{k+1} = \frac{b_j - a_{j1}x_1^{k+1} - ... - a_{jj-1}x_{j-1}^{k+1} - a_{jj+1}x_{j+1}^k - ... - a_{jN}x_N^k}{a_{jj}}$$

for $k = 1, 2, ..., N$

```
function X=jacobi(A,B,P,delta, max1)

% Input  - A is an N x N nonsingular matrix
%        - B is an N x 1 matrix
%        - P is an N x 1 matrix; the initial guess
%       - delta is the tolerance for P
%       - max1 is the maximum number of iterations
% Output - X is an N x 1 matrix: the jacobi approximation to
%         the solution of AX = B

N = length(B);

for k = 1:max1
    for j = 1:N
        X(j)=(B(j)-A(j,[1:j-1,j+1:N])*P([1:j-1,j+1:N]))/A(j,j);
    end
    err = abs(norm(X'-P));
    relerr = err/(norm(X)+eps);
    P = X';
    if (err<delta)|(relerr<delta)
        break
    end
end

X=X';
```

```
function X=gseid(A,B,P,delta, max1)

% Input     - A is an N x N nonsingular matrix
%           - B is an N x 1 matrix
%           - P is an N x 1 matrix; the initial guess
%         - delta is the tolerance for P
%         - max1 is the maximum number of iterations
% Output    - X is an N x 1 matrix: the gauss-seidel approximation to
%           the solution of AX = B

N = length(B);

for k = 1:max1
    for j = 1:N
        if j == 1
            X(1) = (B(1)-A(1,2:N)*P(2:N))/A(1,1);
        elseif j == N
            X(N) = (B(N)-A(N,1:N-1)*(X(1:N-1))')/A(N,N);
        else
            %X contains the kth approximations and P the (k-1)st
            X(j) = (B(j)-A(j,1:j-1)*X(1:j-1)'-A(j,j+1:N)*P(j+1:N))/A(j,j);
        end
    end
    err = abs(norm(X'-P));
    relerr = err/(norm(X)+eps);
    P = X';
    if (err<delta)|(relerr<delta)
        break
    end
end

X=X';
```