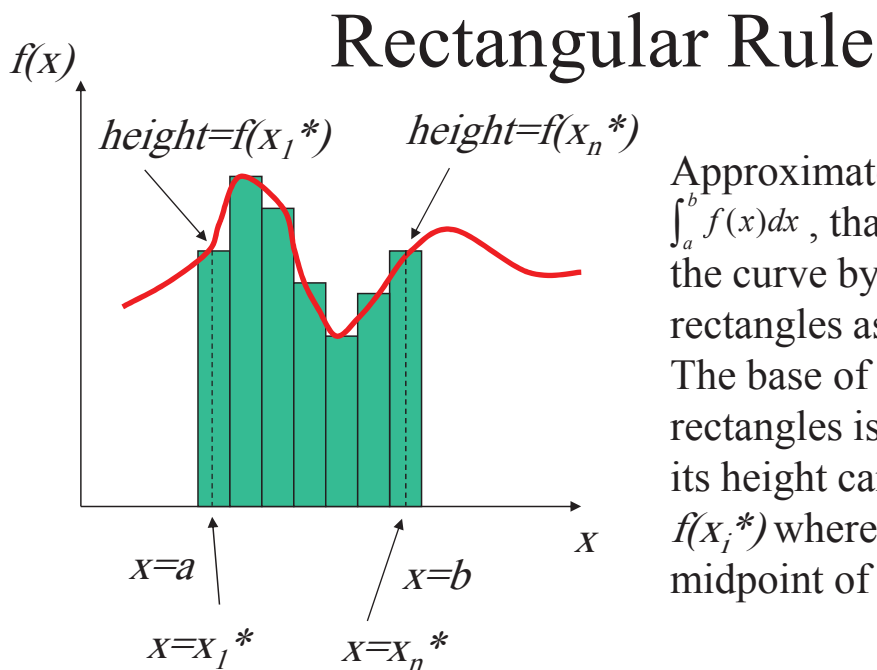


Chapter 7: Numerical Integration

- In general, a numerical integration is the approximation of a definite integration by a “weighted” sum of function values at discretized points within the interval of integration.

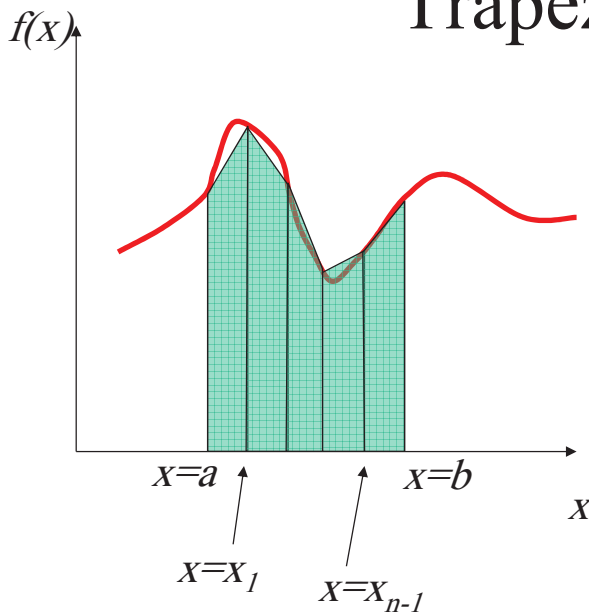
$$\int_a^b f(x)dx \approx \sum_{i=0}^N w_i f(x_i)$$

where w_i is the weighted factor depending on the integration schemes used, and $f(x_i)$ is the function value evaluated at the given point x_i



$$\begin{aligned}\int_a^b f(x)dx &= f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x \\ &= \Delta x[f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)]\end{aligned}$$

Trapezoidal Rule



The rectangular rule can be made more accurate by using trapezoids to replace the rectangles as shown. A linear approximation of the function locally sometimes work much better than using the averaged value like the rectangular rule does.

$$\int_a^b f(x)dx = \frac{\Delta x}{2}[f(a) + f(x_1)] + \frac{\Delta x}{2}[f(x_1) + f(x_2)] + \dots + \frac{\Delta x}{2}[f(x_{n-1}) + f(b)]$$
$$= \Delta x \left[\frac{1}{2} f(a) + f(x_1) + \dots + f(x_{n-1}) + \frac{1}{2} f(b) \right]$$

Simpson's Rule

Still, the more accurate integration formula can be achieved by approximating the local curve by a higher order function, such as a quadratic polynomial. This leads to the Simpson's rule and the formula is given as:

$$\int_a^b f(x)dx = \frac{\Delta x}{3} [f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots$$
$$\dots + 2f(x_{2m-2}) + 4f(x_{2m-1}) + f(b)]$$

It is to be noted that the total number of subdivisions has to be an even number in order for the Simpson's formula to work properly.

Examples

Integrate $f(x) = x^3$ between $x = 1$ and $x = 2$.

$$\int_1^2 x^3 dx = \frac{1}{4} x^4 \Big|_1^2 = \frac{1}{4} (2^4 - 1^4) = 3.75$$

Using 4 subdivisions for the numerical integration: $\Delta x = \frac{2-1}{4} = 0.25$

Rectangular rule:

i	x_i^*	$f(x_i^*)$
1	1.125	1.42
2	1.375	2.60
3	1.625	4.29
4	1.875	6.59

$$\begin{aligned} \int_1^2 x^3 dx &= \Delta x [f(1.125) + f(1.375) + f(1.625) + f(1.875)] \\ &= 0.25(14.9) = 3.725 \end{aligned}$$

Trapezoidal Rule

i	x_i	$f(x_i)$
	1	1
1	1.25	1.95
2	1.5	3.38
3	1.75	5.36
	2	8

$$\begin{aligned} \int_1^2 x^3 dx &= \Delta x \left[\frac{1}{2} f(1) + f(1.25) + f(1.5) + f(1.75) + \frac{1}{2} f(2) \right] \\ &= 0.25(15.19) = 3.80 \end{aligned}$$

Simpson's Rule

$$\begin{aligned} \int_1^2 x^3 dx &= \frac{\Delta x}{3} [f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)] \\ &= \frac{0.25}{3} (45) = 3.75 \Rightarrow \text{perfect estimation} \end{aligned}$$