King Abdullah II School for Information Technology
Department of Computer Science
Numerical Analysis - Midterm - Spring 2016

Student Name: Student ID: Section:

Q1 (X Points) Consider the lower-triangular linear system $AX = B$

$$\begin{aligned} 2x_1 &= 6 \\ -x_1 + 4x_2 &= 5 \\ 3x_1 - 2x_2 - x_3 &= 4 \end{aligned}$$

(a) Use back substitution to solve the system

$2x_1 = 6 \quad \Rightarrow x_1 = 3$
$-3 + 4x_2 = 5 \quad \Rightarrow 4x_2 = 8 \quad \Rightarrow x_2 = 2$
$9 - 4 - x_3 = 4 \quad \Rightarrow x_3 = 1$

(b) Compute det(A)

$det(A) = 2 * 4 * -1 = -8$

Q2 (X Points) If A and B are nonsingular NxN matrices and $C = AB$, show that $C^{-1} = B^{-1}A^{-1}$.

$C^{-1} = B^{-1}A^{-1}$
Multiply both sides by C $\quad \Rightarrow CC^{-1} = CB^{-1}A^{-1}$
Substitute $C = AB$ $\quad \Rightarrow CC^{-1} = ABB^{-1}A^{-1}$
Substitute $BB^{-1} = I$ $\quad \Rightarrow CC^{-1} = A(BB^{-1})A^{-1} \Rightarrow CC^{-1} = AIA^{-1} \Rightarrow CC^{-1} = AA^{-1}$
Substitute $AA^{-1} = I$ $\quad \Rightarrow CC^{-1} = I.$

Q3 (X Points) Which of the following iteration methods will converge to the solution $P = 1$

(a) $x_{n+1} = x_n - \frac{x_n^4 - 1}{4}$
$g(x) = x - \frac{x^4-1}{4}$
$g'(x) = 1 - x^3$
$g'(1) = 1 - 1^3 = 0 < 1$
Sequence will converge

(b) $x_{n+1} = x_n + x_n^4 - 1$
$g(x) = x + x^4 - 1$
$g'(x) = 1 + 4x^3$
$g'(1) = 1 + 4 * 1^3 = 5 > 1$
Sequence will NOT converge

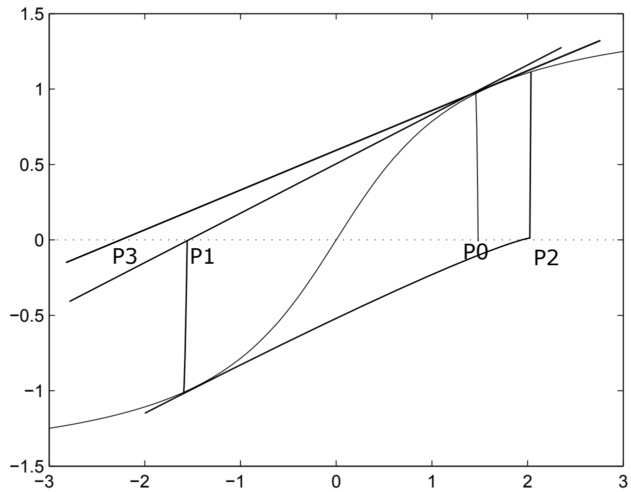
Q5 (X Points) Assume A is a positive number, show that the Nth root of A can be approximated using the recursive formula:

$$p_k = \frac{(N-1)p_{k-1} + \frac{A}{p_{k-1}^{N-1}}}{N} \text{ for } k = 1, 2, ...$$

Start with the function $f(x) = x^N - A$
The roots of the equation $x^N - A = 0$ are $\pm \sqrt[N]{A}$
Newton-Raphson iteration formula:
$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^N - A}{Nx^{N-1}}$
$g(x) = \frac{x(Nx^{N-1}) - x^N - A}{Nx^{N-1}}$
$g(x) = \frac{Nx^N - x^N - A}{Nx^{N-1}}$
$g(x) = \frac{N-1x^N - A}{Nx^{N-1}}$
$g(x) = \frac{x^{N-1}(N-1x - \frac{A}{x^{N-1}})}{Nx^{N-1}}$

$$p_k = \frac{(N-1)p_{k-1} + \frac{A}{p_{k-1}^{N-1}}}{N} \text{ for } k = 1, 2, ...$$

Q4 (X Points) Given the function shown in the graph:



On the graph, illustrate how Newton-Raphson method locates p_1, p_2, and p_3 starting with $p_0 = 1.5$.

Q6 (X Points) Let $f(x) = \sqrt{\pi x} - \cos(\pi x)$.

x	0	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1
$f(x)$	-1	-0.297	0.179	0.702	1.253	1.783	2.242	2.581	2.772

(a) Prove that the equation $f(x) = 0$ has at least a solution p in the interval [0, 1].
Because $f(0) = -1 < 0$ and $f(1) = 2.772 > 0$ there must be at least a root in the interval [0, 1].

(b) By the Bisection method, find c_n, $n \leq 2$ on [0, 1]. Use the table above and write your answers in the next table

n	a_n	b_n	c_n	$f(c_n)$
0	0	1	0.500	1.253
1	0	0.500	0.250	0.179
2	0	0.250	0.125	-0.297

(c) How many iterations are necessary to solve $\sqrt{\pi x} - \cos(\pi x) = 0$ with accuracy 10^{-5} on [0, 1].
$|p_n - P| = \frac{b-a}{2^n}$

$10^{-5} = \frac{1-0}{2^n} \Rightarrow 2^n = \frac{1}{10^{-5}} \Rightarrow 2^n = 10^5 \Rightarrow \log_2 2^n = \log_2 10^5 \Rightarrow n \approx \log_2 2^{17} \Rightarrow n \approx 17$